An-Najah National University

Faculty of Graduate Studies

Analytical and Numerical Methods for Solving Linear Fuzzy Volterra Integral Equation of the Second Kind

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Dedication

I dedicate this thesis to my beloved homeland, to my parents, to my dear husband Sami, to my daughter Shatha, to my sons Hothayfa and yaman, to my sisters specially Malaak and my brothers, to my friend Aminah Alqub , to my school and my students, to everyone who supports and encourages me.

Acknowledgement

First I thank God to complete this thesis, and then extend my sincere thanks to my supervisor Prof. Naji Qatanani who encouraged me all the time and gifted me a lot of self- confidence, and I owe to his effort a great deal because he helped me a lot to achieve my goal in this thesis. Also, l have to thank my family who supported and encouraged me.

Finally, I would like to acknowledge my teachers at Al-Najah National University- Mathematical Department.

الاقرار

أنا الموقع أدناه مقدم الرسالة التي تحمل العنوان:

Analytical and Numerical Methods for Solving Linear Volterra Integral Equation of the Second kind.

أقر بان ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الإشارة اليه حيثما ورد، و أن هذه الرسالة ككل، أو أي جزء منها لم يقدم من قبل لنيل أي درجة علمية أو بحثي لدى أي مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the research's own work, and has not been submitted elsewhere for any other degree or qualification.

Student's name اسم الطالب: جيهان عشيه عبد الرحيم حمايدي

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Date: التاريخ: ٢٤/٥/٢٠١٦م

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Analytical and Numerical Methods for Solving Linear Fuzzy Volterra Integral Equation of the Second Kind

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Abstract

Integral equations, in general, play a very important role in Engineering and technology due to their wide range of applications. Fuzzy Volterra integral equations in particular have many applications such as fuzzy control, fuzzy finance and economic systems.

After introducing some definitions in fuzzy mathematics, we focus our attention on the analytical and numerical methods for solving the fuzzy Volterra integral equation of the second kind.

For the analytical solution of the fuzzy Volterra integral equation we have presented the following methods:

The Fuzzy Laplace Transformation Method(FLTM), Fuzzy Homotopy Analysis Method(FHAM), Fuzzy Adomian Decomposition Method (FADM), Fuzzy Differential Transformation Method (FDTM), and the Fuzzy Successive Approximation Method (FSAM).

For the numerical handling of the fuzzy Volterra Integral equation we have implemented various techniques, namely: Taylor expansion method, Trapezoidal method, and the variation iteration method.

To investigate the efficiency of these numerical techniques we have solved some numerical examples.

Numerical results have shown to be in a close agreement with the analytical ones.

Moreover, the variation iteration method is one of the most powerful numerical techniques for solving Fuzzy Volterra integral equation of the second kind in comparison with other numerical techniques.

Introduction

Fuzzy integral equations of the second kind have attracted the attention of many scientists and researchers in recent years, due to their importance in applications, such as fuzzy control, fuzzy finance, approximate reasoning and economic systems [5].

Prior to discussing fuzzy differential equations and integral equations and their associated numerical algorithms, it is necessary to present an appropriate brief introduction to preliminary topics, such as fuzzy numbers and fuzzy calculus. The concept of fuzzy sets, was originally introduced by Zadeh [48], led to the definition of fuzzy numbers and its implementation in fuzzy control [17], and approximate reasoning problems [49].

The concept of integral of fuzzy functions was first introduced by Dubios and Prade [21]. Alternative approaches were later suggested by Goetschel and Voxman [30], Kaleva [33], Nanda [40] and others. One of the first applications of fuzzy integration was given by Wu and Ma [18] who investigated the fuzzy Fredholm integral equation of the second kind.

Liao [34] in 1992 employed the homotopy analysis method to solve non-linear problems. In addition, the homotopy analysis method has been used for solving fuzzy integral equations of the second kind. Babolian etal. [9] have solved the fuzzy integral equation by the Adomian method. There are also numerous numerical methods which have been focusing on the solution of integral equations [23]. Amawi [6] has investigated some analytical and numerical methods for solving fuzzy Fredholm integral equation of the second kind. Amawi and Qatanani [7] have employed the Taylor expansion method, the Trapezoidal rule, the Adomian method and the homotopy analysis method to solve various fuzzy Fredholm integral equations of the second kind.

This thesis is organized as follows:

In chapter one, we introduce some basic concepts of fuzzy sets, crisp sets, fuzzy numbers, and fuzzy integral equation.

In chapter two, we investigate some analytical methods to solve linear fuzzy Volterra integral equation of the second kind, namely: Fuzzy Laplace transformation method(FLTM), Fuzzy Homotopy analysis method(FHAM), Fuzzy Adomian decomposition method(FADM), and Fuzzy successive approximation method.

In chapter three, we employ some numerical methods to solve linear fuzzy Volterra integral equation of the second kind, these include:

Fuzzy Taylor expansion method, Fuzzy Trapezoidal method, and Fuzzy variational iteration method.

In chapter four, MAPLE software has been constructed to solve numerical examples to demonstrate the efficiency of these numerical schemes introduced in chapter three.

Finally, we draw a comparison between analytical and numerical solutions for some numerical examples.

Chapter One

Mathematical Preliminaries

Basic Definitions and Operations.

1.1 Crisp and Fuzzy Sets

Definition (1.1) [14]: Characteristic function: Let X be a set and A be a subset of X ($A \subseteq X$). Then the characteristic function of the set A in X, where $\boldsymbol{\mu_A} : X \rightarrow \{0,1\}$ is defined by:

$$\boldsymbol{\mu_{A(x)}} = \begin{cases} 1 & \text{if } x \in \text{A} \\ 0 & \text{if } x \notin \text{A} \end{cases}$$

Definition (1.2) [48]: Fuzzy set: a Fuzzy set is a set whose element have degrees of membership.

It is extension of the classical notion of set.

The value 0 is used to represent complete non-membership, the value 1 is used to represent complete membership, and values in between are used to represent intermediate degrees of membership.

Definition (1.3) [12]: The support of fuzzy set: The support of a fuzzy set A is defined by :

$$Supp(A) = \{x \epsilon\, X : A(x) > 0\}.$$

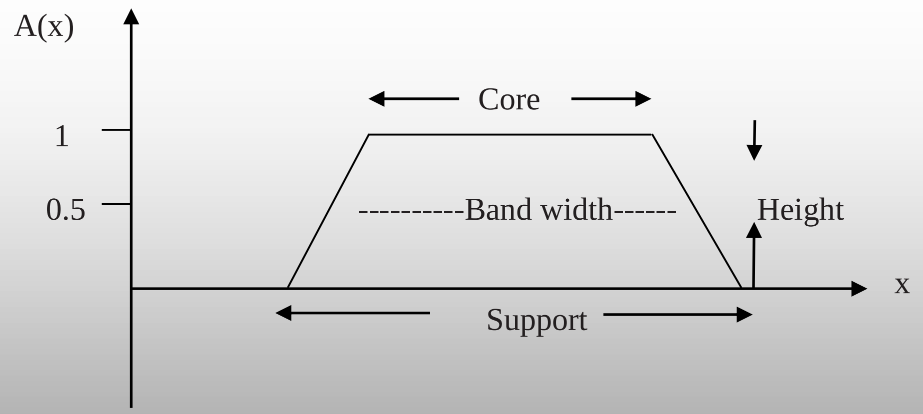


Figure (1.1)

$Core\ of A\ = \{x: A(x)\ =\ 1\}$

$Support\ of\ A\ =\ \{x: A(x)\ >\ 0\}$

$Band\ width\ of\ \ A\ = \{x: A(x)\ \geq\ 0.5\}$

$Height\ of\ A\ =\ \sup_{x\in X} A(x)$

Definition(1.4) [25]: $\boldsymbol{\alpha}$ – cut: An α-level set of a fuzzy set A of X is a non-fuzzy set denoted by $[A]^{\alpha}$ and is defined by**:**

$$[A]^{\alpha} = \begin{cases} x \in X: A(x) \geq \alpha & , if \alpha > 0 \\ cl(supp(A)) & , if\ \alpha = 0 \end{cases}$$

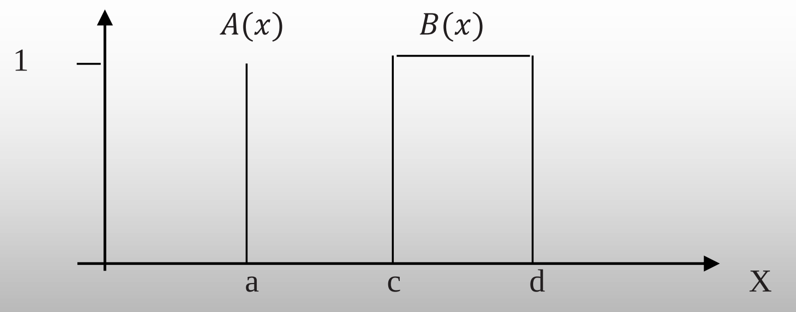
where $cl\ (supp(A))\ =\ Closure\ of\ the\ support\ A$.

Definition (1.5) [25]: Crisp number: A crisp number a is represented by:

$$A(x) = \begin{cases} 1 & if\ x\ =\ a \\ 0 & if\ x\ \neq\ a \end{cases}$$

a crisp interval $[c, d]$ is represented by a fuzzy set:

$$B(x) = \begin{cases} 1 & if\ x \in [c, d] \\ 0 & if\ x \notin [c, d] \end{cases}$$



The crisp number

$Figure$ (1.2)

1.2 Operations on Fuzzy Sets

As in the case of ordinary sets, the operations on fuzzy sets play a control role in the case of fuzzy sets, Zadeh [48] defined the following operations for fuzzy sets as generalization of crisp sets and of crisp statements.

Definition (1.6): The intersection: the membership function of the intersection of two fuzzy sets A and B is defined as:

$$(A \cap B)\ (x) \ = \ Min\ \{A(x), B(x)\}, \forall\ x \in X.$$

or, in abbreviated form: $(A \cap B)\ (x) \ = \ A\ (x) \wedge B(x)$

Definition (1.7): The union: the membership function of the union of two fuzzy sets A and B is defined as: $(A \cup B)\ (x) \ = \ Max\ \{A(x), B(x)\}, \forall x \in X$.

or, in abbreviated form: $(A \cup B)\ (x) = \ A\ (x) \vee B(x)$

Definition (1.8): The complement: the membership function of complement is defined as : $A(x)^c = \ 1 - \ A(x), for\ every\ x \epsilon\ X$.

Definition (1.9): The containment: A fuzzy set A is contained in a fuzzy set B $(A \subseteq B)$ if and only if $A\ (x) \ \le B(x),\ \forall x \in X$.

In symbols, $A \ \subseteq B \ \Leftrightarrow \ A\ (x) \ \le B(x)$.

Definition (1.10): The equality: The two fuzzy sets A and B are equal if and only if

$A\ (x) \ = B(x), \forall x \in X$.

In symbols, $A \ = B \ \Leftrightarrow \ A\ (x) = B(x), \forall x \in X$.

Definition (1.11): The empty fuzzy set: A fuzzy set A is empty if and only if its membership function $A(x) \ = \ 0,\ \forall x \in X$.

Some Properties of Fuzzy Sets

$$\left.\begin{array}{l} \bullet\ (A \cup B)^C = A^C \cap B^C \\[2ex] \bullet\ (A \cap B)^C = A^C \cup B^C \end{array}\right\} \quad \text{De Morgan's laws}$$

$$\left.\begin{array}{l} \bullet\ C \cap (A \cup B) = (C \cap A) \cup (C \cap A) \\ \bullet\ C \cup (A \cap B) = (C \cup A) \cap (C \cup A) \end{array}\right\} \text{Distributive laws}$$

- $1-\text{Max}\{A(x), B(x)\} = \text{Min}\ \{1 - A(x), 1 - B(x)\}$

- $1-\text{Min}\{A(x), B(x)\} = \text{Max}\ \{1 - A(x), 1 - B(x)\}$

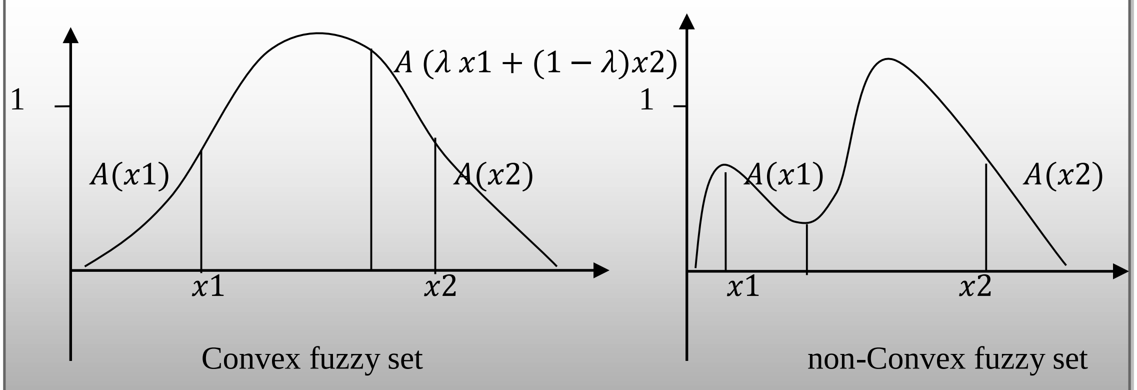
1.3 Fuzzy Numbers

Definition (1.12) [48]: Normal fuzzy set: A fuzzy set A is called normal if there exists $an\ element\ x \in X\ such\ that\ A(x) = 1$, Otherwise A is subnormal.

Definition (1.12) [4]: Convex fuzzy set: A fuzzy set A is convex if**:**

$A\ (\lambda x_1 + (1 - \lambda)\ x_2\ ) \geq\ min\ \{A\ (x_1), A\ (x_2)\}, \forall\ x_1, x_2 \epsilon\ X\ , and$

$\lambda \epsilon$ [0,1]. A is convex if all its $\alpha - cuts$ are convex.



$Figure$ (1.3)

Definition (1.13) [47]: Upper semi continuous: A function f: $E \longrightarrow R$ is said to be upper semi continuous at $x_0 \in E$ if :

$\forall \varepsilon > 0, \exists \delta > 0\ s.t\ f(x) = f(x_0) + \varepsilon$, Where $x \in E \cap B_\delta(x_0)$.

Definition (1.14) [28]: Fuzzy number: A fuzzy number is a fuzzy set

$A: R \rightarrow [0,1]$ such that:

(i) A is upper semi continuous,

(ii) $A(x) = 0$ outside some interval $[a, d]$,

(iii) There are real numbers $b, c: a \leq b \leq c \leq d$, for which:

(1) $A(x)$ is monotonically increasing on $[a, b]$,

(2) $A(x)$ is monotonically decreasing on $[c, d]$,

(3) $A(x) = 1, b \leq x \leq c$.

Definition (1.15) [50]: Convex normalized fuzzy set: A fuzzy number A is a convex normalized if:

1. there exists exactly one $x_0 \in R: A(x_0) = 1$ (x_0 is called the mean value of A)
2. $A(x)$ is piecewise continuous.

Definition (1.16) [50]: The cardinality: For finite fuzzy set A, the cardinality of A is defined as :

$$|\boldsymbol{A}| = \sum_{x \in X} \boldsymbol{A}(\boldsymbol{x})$$

$\|\boldsymbol{A}\| = \frac{|\boldsymbol{A}|}{|\boldsymbol{X}|}$ is called the relative cardinality of A .

Definition (1.17) [24]: Triangular fuzzy number: Triangular a fuzzy number represented with three points as follows $A = (\, a_1, a_2, a_3\,)$

This representation is interpreted as membership functions and holds the following conditions:

(i) a_1 to a_2 is increasing function.

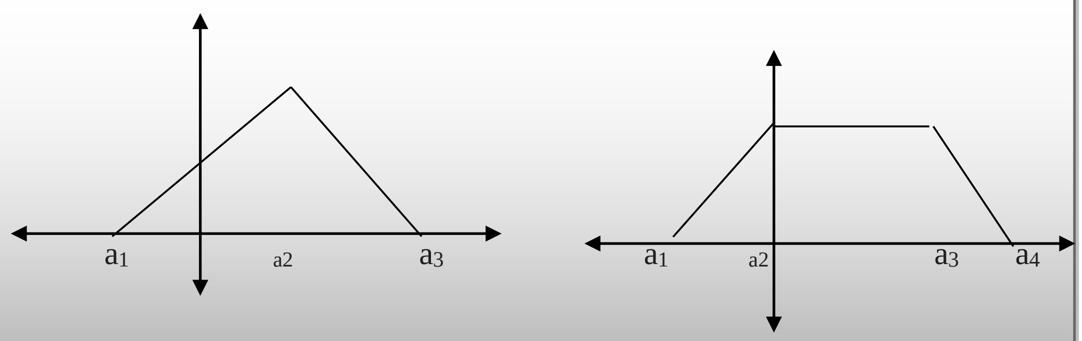
(ii) a_2 to $a_3$is decreasing function.

$(iii) a_1 \leq a_2 \leq a_3$.

$$A(x) = \begin{cases} 0 & for\; x < a_1 \\ \dfrac{x - a_1}{a_2 - a_1} & if\; a_1 \leq x \leq a_2 \\ \dfrac{a_3 - x}{a_3 - a_2} & if\; a_2 \leq x \leq a_3 \\ 0 & for\; x > a_3 \end{cases}$$

Definition (1.18) [10]: Trapezoidal fuzzy number: We can define trapezoidal fuzzy number A as $A = (\, a_1, a_2, a_3, a_4\,)$The membership of this fuzzy number will be interpreted as follows:

$$A(x) = \begin{cases} 0 & for\; x < a_1 \\ \dfrac{x - a_2}{a_2 - a_1} & if\; a_1 \leq x \leq a_2 \\ 1 & if\; a_2 \leq x \leq a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & if\; a_3 \leq x \leq a_4 \\ 0 & for\; x > a_4 \end{cases}$$



Triangular fuzzy number Trapezoidal fuzzy number

Figure (1.4)

Definition (1.19): Fuzzy metric: Given two fuzzy sets A, B. we try to find a fuzzy number d*(A, B) that should satisfy the following:

(i) $d^*(A,B) \geq 0, and \;\; d^*(A,B) \Leftrightarrow A = B$

(ii) $d^*(A,B) = d^*(B,A)$

(iii) $d^*(A,C) \leq d^*(A,B) + d^*(B,C), for\ any\ sets\ A, B, C$

Definition (1.20) [2]: Parametric form of fuzzy number:

A fuzzy number x is a pair $(\underline{x}, \bar{x})$ of functions $\underline{x}\ (\alpha), \bar{x}(\ \alpha);\ 0 \leq \alpha \leq 1$ which satisfies the following:

$i.$ $\underline{x}(\alpha)$is a bounded left-continuous non-decreasing function over [0, 1]

$ii.$ $\overline{x}(\alpha)$is a bounded left-continuous non-increasing function over [0, 1]

$iii.$ $\underline{x}\ (\alpha) \leq \overline{x}\ (\alpha);\ 0 \leq \alpha \leq 1$

A crisp number α is simply represented by $\underline{x}(\alpha) = \bar{x}(\alpha) = \alpha$,

$0 \leq \alpha \leq 1$.

arithmetic operations of arbitrary fuzzy numbers $x = (\underline{x}, \bar{x}), y = (\underline{y}, \bar{y})$, and $\lambda \in R$, can be defined as **[43]**:

1. $x = y \ iff \ \underline{x}(\alpha) = \underline{y}(\alpha) and \ \overline{x}(\alpha) = \overline{y}(\alpha)$,

2. $x + y = \left[\underline{x}(\alpha) + \underline{y}(\alpha), \overline{x}(\alpha) + \overline{y}(\alpha)\right]$,

3. $x - y = \left[\underline{x}(\alpha) - \underline{y}(\alpha), \overline{x}(\alpha) - \overline{y}(\alpha)\right]$,

4. $\lambda x = \begin{cases} \lambda\underline{x}(\alpha), \lambda\overline{x}(\alpha), \lambda \geq 0 \\ \lambda\overline{x}(\alpha), \lambda\underline{x}(\alpha), \lambda < 0 \end{cases}$

Definition (1.21) [3]: The distance: For arbitrary fuzzy numbers $x = (\underline{x}, \overline{x}) and \ y = (\underline{y}, \overline{y})$ we define the distance between x and y by

$$d(x, y) = \sup_{0 \leq \alpha \leq 1} \left\{\max\left[\left|\underline{x} - \underline{y}\right|, |\overline{x} - \overline{y}|\right]\right\}$$

Let d: $E \times E \longrightarrow R, where \ d \ is \ the \ space \ of \ fuzzy \ number.$

The following are satisfied:

1. $d(x + z, y + z) = d(x, y), \forall x, y, z \in E$.

2. $d(\lambda x, \lambda y) = |\lambda| d(x, y), \lambda \in R, x, y \in E$.

3. $d(x + y, z + w) \leq d(x, z) + d(y, w), \forall x, y, z, w \in E$.

the function $d \ (x \ , y)$ is a metric on E.

1.4 Fuzzy function

Definition (1.22) [42]: Fuzzy function: Let $X\ and\ Y$ are universal sets and $f: X \longrightarrow Y$ be a function. Let moreover $F(X), F(Y)$ be respective universes of fuzzy sets, identified with their membership functions, $i.e\ \ F(X) = \{A: X \to [0,1]\}$ and similarly $F(Y)$. By extension principle, f induces a function $f^*: F(X) \longrightarrow F(Y)$ such that for all $A \subset F(X)$

$$f^*(A)(y) = \begin{cases} 0 & if\ f^{-1}(y) = \emptyset \\ sup\{A(x): x \in f^{-1}(y)\} & if\ f^{-1}(y) \neq \emptyset \end{cases}$$

Definition (1.23) [20]: Continuous fuzzy function

A function $f : [a, b] \to E$ is said to be continuous fuzzy function if for arbitrary fixed $x_0 \in [a, b]$, $\delta > 0$ such that:

$$|x - x_0| < \delta \Longrightarrow d(f(x), f(x_0)) < \varepsilon$$

Definition (1.24) [46]: Differential of a fuzzy function:

Let $u: (a, b) \longrightarrow E$ be a fuzzy function and $x_0 \in (a, b)$. u is differentiable at x_0 if two forms were sustained as follows:

- It exists an element $u'(x_0) \in E$ such that , for all $h > 0$ sufficiently near to 0, there are $u(x_0 + h) - u(x_0), u(x_0) - u(x_0 - h)$and the limits :

$$\lim_{h \to 0+} \frac{u(x_0 + h) - u(x_0)}{h} = \lim_{h \to 0+} \frac{u(x_0) - u(x_0 - h)}{h} = u'(x_0)$$

- It exists an element $u'(x_0) \in E$ such that , for all h < 0 sufficiently near to 0, there are $u(x_0 + h) - u(x_0), u(x_0) - u(x_0 - h)$ and the limits :

$$\lim_{h \to 0-} \frac{u(x_0 + h) - u(x_0)}{h} = \lim_{h \to 0-} \frac{u(x_0) - u(x_0 - h)}{h} = u'(x_0)$$

Let $u: (a, b) \to E$ be a fuzzy function and denote:

$[u(x)]^\alpha = \left[\, \underline{u}(x, \alpha), \overline{u}(x, \alpha)\right], \forall \alpha \in [0, 1]$ and $x \in (0,1)$

If f is differentiable, then $\underline{u}(x, \alpha) and\, \overline{u}(x, \alpha)$are differentiable functions and

$$[u'(x)]^\alpha = \left[\, \underline{u'}(x, \alpha), u(x, \alpha)\right]$$

Definition (1.25) [13]: Strongly generalized differentiable

Let $u: (a, b) \to E$ and $x_0 \in (a, b)$, We say that u is strongly generalized differentiable at x_0 if there exists an element $u'(x_0) \in E$ such that:

(i) $\forall\ h > 0$ Sufficiently small, $\exists u(x_0 + h) - u(x_0), u(x_0) - u(x_0 - h)$ and the limits (in the metric d)

$$\mathop{lim}_{h \to 0} \frac{u(x_0 + h) - u(x_0)}{h} = \mathop{lim}_{h \to 0} \frac{u(x_0) - u(x_0 - h)}{h} = u'(x_0),$$

or

(ii) $\forall\ h > 0$ Sufficiently small, $\exists u(x_0) - u(x_0 + h), u(x_0 - h) - u(x_0)$ and limits

$$\lim_{h\to 0}\frac{u(x_0)-u(x_0+h)}{h}=\lim_{h\to 0}\frac{u(x_0-h)-u(x_0)}{h}=u'(x_0),$$

or

(iii) $\forall\ h>0$ Sufficiently small, $\exists u(x_0+h)-u(x_0), u(x_0-h)-u(x_0)$ and the limits

$$\lim_{h\to 0}\frac{u(x_0)-u(x_0+h)}{-h}=\lim_{h\to 0}\frac{u(x_0)-u(x_0-h)}{-h}=u'(x_0),$$

or

(iv) $\forall\ h>0$ Sufficientlysmall,$\exists u(x_0)-u(x_0+h), u(x_0)-u(x_0-h)$ and the limits

$$\lim_{h\to\to 0}\frac{u(x_0)-u(x_0+h)}{-h}=\lim_{h\to 0}\frac{u(x_0)-u(x_0-h)}{h}=u'(x_0)$$

We will define the integral of a fuzzy function using the Riemann integral concept.

Definition (1.26) [2]: Integral of a fuzzy function: Let $u:\ [a,b]\to\ E$. for each partition $p=\{x_0,x_1,\dots,x_n\}$ of [a, b] and for arbitrary $\xi\ i$:

$x_{i-1}\ \le\ \xi\ i\ \le x_i, 1\ \le\ i\ \le\ n, let\ \lambda\ =\ \max_{1\le i\le n}\{|x_i-x_{i-1}|\}\ and$

$$R_P=\sum_{i=1}^{n}u(\ \xi\ i)(x_i-x_{i-1})$$

the definite integral of $u(x)$ over $[a,b]$ is:

$$\int_a^b u(x)dx=\lim_{\lambda\to 0}R_P$$

provided that this limit exists in the metric d.

If the fuzzy function $u(x)$ is continuous in the metric d, its definite integral exists, furthermore.

$$\underline{\left(\int_a^b u(x,\alpha)dx\right)} = \int_a^b \underline{u}(x,\alpha)dx$$

$$\overline{\left(\int_a^b u(x,\alpha)dx\right)} = \int_a^b \overline{u}(x,\alpha)dx$$

where $\left(\underline{u}(x,\alpha), \overline{u}(x,\alpha)\right)$ is the parametric form of $u(x)$. It should be noted that the fuzzy integral can be also defined using the Lebesgue- type approach. However, if $u\ (x)$ is continuous, both approaches yield the same value. Moreover, the representation of the fuzzy integral is more convenient for numerical calculations.

Properties of the fuzzy integral [22]

Let $F, G: X \longrightarrow E$ be integrable and $\lambda \in R$, then the following are satisfied:

(1) $\int_X (F(x) + G(x))dx = \int_X F(x)dx + \int_X G(x)dx$

(2) $\int_X \lambda F(x)dx = \lambda \int_X F(x)dx$

(3) $d(F,G)$ integrable

(4) $d\left(\int_X F(x)dx\,, \int_X G\ (x)dx\right) \leq \int_X d(F,G)dt$

1.5 Fuzzy Linear Systems

Systems of simulations linear equations play major role in various areas such as mathematics, statistics, and social sciences. Since in many applications, at least some of the system's parameters and measurements are represented by fuzzy rather than crisp numbers, therefore, it's important to develop mathematical models and numerical procedures that would appropriately treat general fuzzy linear systems and solve them.

The system of linear equations $TX = C$ where the coefficient matrix T is crisp, while C is a fuzzy number vector, is called a fuzzy system of linear equations (FSLE).

Definition (1.27) [1]: Fuzzy linear system: The $n \times n$ linear system of equations

$$\begin{array}{l} t_{11}\, x_1 + t_{12}\, x_2 + \cdots + t_{1n} x_n = c_1 \\ t_{21}\, x_1 + t_{22}\, x_2 + \cdots + t_{2n}\, x_n = c_2 \\ \vdots \\ t_{n1}\, x_1 + t_{n2}\, x_2 + \cdots + t_{nn}\, x_n = c_n \end{array} \tag{1.1}$$

or, in matrix form:

$$TX = C \tag{1.2}$$

where the coefficient matrix $T = t_{ij}, 1 \leq i, j \leq n$ is a crisp $n \times n$ matrix and $c_i \in E, 1 \leq i \leq n$ .This system is called a fuzzy linear system (FLS).

Definition (1.28) [3]: Solution of fuzzy linear system:

A fuzzy vector $X = (x_1, x_2, \dots, x_n)^t$, given by $x_i\ (\alpha) = \left[\overline{x}_i(r)\ , \underline{x}_i(r)\right]$, is called the solution of (1.1) if:

$$\underline{\sum_{j=1}^{n} \boldsymbol{t_{ij}\, x_j}} = \sum_{j=1}^{n} \underline{\boldsymbol{t_{ij}\, x_j}} = \underline{\boldsymbol{c}}_{\boldsymbol{i}}, \quad \overline{\sum_{j=1}^{n} \boldsymbol{t_{ij}\, x_j}} = \sum_{j=1}^{n} \overline{\boldsymbol{t_{ij}\, x_j}} = \overline{\boldsymbol{c}}_{\boldsymbol{i}},$$

we introduce the notations below **[8]**:

$$\begin{aligned} x &= \left(\underline{x}_1, \underline{x}_2, \dots, \underline{x}_n, -\overline{x}_1, -\overline{x}_2, \dots, -\overline{x}_n\right)^t = \left(\underline{x}, -\overline{x}\right)^t \\ c &= \left(c_1, \underline{c}_2, \dots, \underline{c}_n, -\overline{c}_1, -\overline{c}_2, \dots, -\overline{c}_n\right)^t = \left(\underline{c}, -\overline{c}\right)^t \end{aligned} \tag{1.3}$$

$Z = \left(z_{ij}\right), 1 \leq i, j \leq 2n, where\ z_{ij}$ are determined as follows:

$t_{ij} \geq 0 \Longrightarrow z_{ij} = t_{ij}, z_{i+n,j+n} = t_{ij},$

$t_{ij} < 0 \Longrightarrow z_{i,j+\,n} = -t_{ij}, z_{i+n,j} = -t_{ij}$

and any z_{ij} which is not determined by (1.3) is zero. Using matrix notation we have

$$ZX = C \tag{1.4}$$

the structure of Z implies that $z_{ij} \geq 0$ and that

$$Z = \begin{bmatrix} M & W \\ W & M \end{bmatrix} \tag{1.5}$$

where M contains the positive elements of T, W contains the absolute value of the negative elements of T and $T = M - W$.

If matrix T is nonsingular, then matrix Z may be singular.

Theorem (1.2) [40]: If Z^{-1}exists it must have the same structure as $Z, i.e.$

$$Z^{-1} = \begin{pmatrix} D & E \\ E & D \end{pmatrix} \tag{1.6}$$

assuming that Z is nonsingular we obtain:

$$X = Z^{-1}\,C \tag{1.7}$$

Proof: see [40]

Theorem (1.3) [43]: The unique solution X of equation (1.7) is a fuzzy vector for arbitrary b if and only if Z^{-1} is nonnegative, $i.e.$

$$(Z^{-1})_{ij} \ge 0, \qquad 1 \le i,j \le 2n \tag{1.8}$$

Proof: see [43]

Definition (1.29) [2]: Strong fuzzy solution

Let $X = \{(\underline{x}_i(\alpha), \overline{x}_i(\alpha)), 1 \le i \le n\}$ denotes the unique solution of (1.1). the fuzzy number vector $g = \left\{\left(\underline{g}_i(\alpha), \overline{g}_i(\alpha)\right), 1 \le i \le n\right\}$ defined by:

$$\underline{g}_i(\alpha) = min\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), x_i(1)\}$$

(1.9)

$$\overline{g}_i(\alpha) = \max\{\underline{x}_i(\alpha), \overline{x}_i(\alpha), x_i(1)\}$$

is called the fuzzy solution of $ZX = C$. Moreover if

$\left(\underline{x}_i(\alpha), \overline{x}_i(\alpha)\right), 1 \leq i \leq n,$ are all of fuzzy numbers then we have $\underline{x}_i(\alpha) = \underline{g}_i(r), \overline{x}_i(\alpha) = \overline{g}_i(\alpha), 1 \leq i \leq n,$ and g is called a strong fuzzy solution. Otherwise, g is a week fuzzy solution.

necessary and sufficient conditions for the existence of a strong solution can be described by the following theorem:

Theorem (1.4) [8]:

Let $Z = \begin{bmatrix} M & W \\ W & M \end{bmatrix}$ be a nonsingular matrix. The system (1.4) has a strong solution if and only if:

$$(M + W)^{-1}\left(\underline{c} - \overline{c}\right) \leq 0 \qquad (1.10)$$

Proof: see [8]

Theorem (1.5) [8]: The (FLS) (1.1) has a unique strong solution if and only if the following conditions hold:

1) The matrices $T = M - W \ and \ M + W$ $a$re both nonsingulars.
2) $(M + W)^{-1}(c - \overline{c}) \leq 0$.

1.6 Fuzzy integral equations

Definition (1.30): Integral equation: An integral equation is the equation in which the unknown function appears under an integral sign.

In this section, the fuzzy integral equation of second kind is introduced.

the Fredholm integral equation of the second kind is given by **[32]:**

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)dt \qquad (1.11)$$

where $\lambda > 0, k(x,t)$is an arbitrary kernal function over the square

$a \leq t, x \leq b$ and $f(x)$is a function of x: $a \leq x \leq b$.

If the kernel function satisfies $k(x,t) = 0$, t $> x$, we obtain the Volterra integral equation:

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt \qquad (1.12)$$

if $f\ (x)$ is a fuzzy function, we have Volterra fuzzy integral of the second kind.

Definition (1.30) [27]: General form of Volterra integral equation of the second kind: The second kind fuzzy Volterra integral equation system is in the form

$$u_i(x) = f_i(x) + \sum_{j=1}^{m} \lambda_{ij} \int_a^x k_{1j}(x,t)u_j(t)dt \qquad (1.13)$$

$a \leq t \leq x \leq b\ and\ \lambda_{ij} \neq 0\ (for\ i,j = 1,2,3,\dots,m)$ are real constants. Moreover, in system (1.13), the fuzzy function $f_i(x)$ and kernel $k\_ij\ (x,t)$ are given and assumed to be sufficiently differentiable with respect to all their arguments on the interval $a \leq t, x \leq b$, and we assume that the kernel function $k_{ij}(x,t) \in L^2([a,b] \times [a,b])$, and

$u(x) = [u_1, u_2, \dots, u_m]^t$is the solution to be determined.

now,$\left(\underline{f_i}(x,\alpha), \overline{f}_i(x,\alpha)\right)$and $\left(\underline{u}_i(x,\alpha), \overline{u}_i(x,\alpha)\right)$, $(0 \leq \alpha \leq 1, a \leq x \leq b)$

be parametric form of $f_i(x)$ and $u_i(x)$, respectively. To simplify, we assume that $\lambda_{ij} > 0 (for\ i,j = 1,2,\dots,m)$.

In order to design a numerical scheme for solving (1.16), we write the parametric form of the given fuzzy Volterra integral equations system as follows:

$$\overline{u}_i(x,\alpha) = \overline{f}_i(x,\alpha) + \sum_{j=1}^{m}\left(\lambda_{ij}\int_a^x \overline{g}_{ij}(t,\alpha)dt\right)$$

$$i = 1,2,\dots,m \qquad (1.14)$$

$$\underline{u}_i(x,\alpha) = \underline{f}_i(x,\alpha) + \sum_{j=1}^{m}\left(\lambda_{ij}\int_a^x \underline{g}_{ij}(t,\alpha)dt\right)$$

where

$$\overline{g}_{ij}(t,\alpha) = \begin{cases} k_{ij}(x,t)\overline{u}_j(t,\alpha), & k_{ij}(x,t) \geq 0 \\ k_{ij}(x,t)\underline{u}_j(t,\alpha), & k_{ij}(x,t) < 0 \end{cases} \qquad (1.15)$$

and

$$\underline{g}_{ij}(t,\alpha) = \begin{cases} k_{ij}(x,t)\underline{u}_j(t,\alpha), & k_{ij}(x,t) \geq 0 \\ k_{ij}(x,t)\overline{u}_j(t,\alpha), & k_{ij}(x,t) < 0 \end{cases} \qquad (1.16)$$

Fuzzy integro-differential equations

The linear fuzzy integro-differential equation is given by **[38]:**

$$u'(x) = f(x) + \lambda \int_a^{h(x)} k(x,t)u(t)dt\,, u(x_0) = u_0 \qquad (1.17)$$

where $\lambda > 0$, the kernel function $k(x,t) \in L^2\,[a,b], a \leq t, x \leq b,$ and $f(x)$is a given function of $x \in [a,b]$.If u is a fuzzy function, $f(x)$is a given fuzzy function of $x \in [a,b]$ and u' is the first fuzzy derivative of u. This equation may be possessing fuzzy solution.

If the kernel function satisfies $k(\,x,t\,) \,= 0, h(x)$is a variable, we obtain the Voltera integro-differential equation:

$$u'^{(x)} = f(x) + \lambda \int_a^x k(x,t)u(t)dt \tag{1.18}$$

If $h(x) = b$ is a constant then the equation (1.20) is fuzzy Fredholm integro-differential equation of the second kind is given by:

$$u'^{(x)} = f(x) + \lambda \int_a^b k(x,t)u(t)dt \tag{1.19}$$

Let $u(x) = \left(\underline{u}(x,\alpha), \overline{u}(x,\alpha)\right)$ is a fuzzy solution of equation (1.17). using definitions (1.17), and (1.18) we have the equivalent system:

$$\begin{cases} \underline{u'}(x) = \underline{f}(x) + \lambda \int_a^x \underline{g}(t,\alpha)dt, \underline{u}(x_0) = \underline{u}_0 \\ \overline{u'}(x) = \overline{f}(x) + \lambda \int_a^x \overline{g}(t,\alpha)dt, \overline{u}(x_0) = \overline{u}_0 \end{cases} \tag{1.20}$$

for each $0 \leq \alpha \leq 1 and\ s, x \in [a,b]$, where

$$\underline{g}(t,\alpha) = \begin{cases} k(x,t)\underline{u}(t,\alpha), & k(x,t) \geq 0 \\ k(x,t)\overline{u}(t,\alpha), & k(x,t) < 0 \end{cases}$$

and (1.21)

$$\overline{g}(t,\alpha) = \begin{cases} k(s,t)\overline{u}(t,\alpha), & k(x,t) \geq 0 \\ k(s,t)\underline{u}(t,\alpha), & k(x,t) < 0 \end{cases}$$

this possesses a unique solution. $\left(\overline{u}, \underline{u}\right) \in B$

which is a fuzzy function, $i.e. for\ each\ x$, $\left(\overline{u}(x,\alpha), \underline{u}(x,\alpha)\right)$ is a fuzzy number, therefore each solution of equation(1.17) is a solution of system (1.20) and conversely.

Chapter Two

Analytical Methods For Solving Fuzzy Volterra Integral Equation of the second Kind

In this chapter we will investigate analytical solutions for the linear Volterra fuzzy integral equation of the second kind , namely; fuzzy Laplace transform method, homotopy analysis method, the Adomian decomposition method and differential transformation method.

2.1 Fuzzy Laplace Transformation Method

In this section, we will introduce some basic definitions for the fuzzy Laplace transform and fuzzy convolution, then solve fuzzy convolution Volterra integral equation of the second kind by using fuzzy Laplace transform method.

Theorem (2.1) [**19**]: Suppose that $u(x)$ is a fuzzy valued function on[$a, \infty)$ represented by the parametric form $\left(\underline{u}(x,\alpha), \overline{u}(x,\alpha)\right)$, for any number $\alpha \in [0,1]$. Assume that $\underline{u}(x,\alpha)\ and\ \overline{u}(x,\alpha)$ are Riemann-Integrable on $[a,b], for\ every\ b \geq a$. Also we assume that $\underline{\varphi}(\alpha)\ and\ \overline{\varphi}(\alpha)$ are positive functions, such that:

$$\int_a^b \left|\underline{u}(x,\alpha)\right| dx \leq \underline{\varphi}(\alpha)\ and \int_a^b |\overline{u}(x,\alpha)|\, dx \leq \overline{\varphi}(\alpha), for\ every\ b \geq a,$$

Then $u(x)$ is improper fuzzy Riemann-Integrable on $[a, \infty)$ which is a fuzzy number, we have:

$$\int_a^b u(x,\alpha)dx = \left(\int_a^b \underline{u}(x,\alpha)dx, \int_a^b \overline{u}(x,\alpha)dx\right)$$

Proposition (2.1) [**15**]: Let $u(x)$ *and* $v(x)$ are two fuzzy valued functions and fuzzy Riemann-integrable on $[a, \infty)$, then $u(x) + v(x)$ is fuzzy Riemann-integrable on $[a, \infty)$.

Moreover, we have:

$$\int_a^\infty (u(x) + v(x))\, dx = \int_a^\infty u(x)\, dx + \int_a^\infty v(x)\, dx$$

Definition (2.1) [19]: Fuzzy Laplace transform:

Let $u(x)$ be a fuzzy valued function and s is a real parameter, then the fuzzy Laplace transform of the function u with respect to t denoted by $U(s)$ is defined by:

$$U(s) = L(u(x)) = \int_0^\infty e^{-sx}u(x)dx = \lim_{\tau\to\infty}\int_0^\tau e^{-sx}u(x)dx, \qquad (2.1)$$

using theorem (2.1), we obtain:

$$U(s) = \left[\lim_{\tau\to\infty}\int_0^\tau e^{-sx}\underline{u}(x)dx\, , \lim_{\tau\to\infty}\int_0^\tau e^{-sx}\,\overline{u}(x)dx\right]$$

where the limits exist.

the $\alpha - cut$ representation of $U(s)$ is given by:

$$U(s,\alpha) = L(u(x,\alpha)) = \left[L\left(\underline{u}(x,\alpha)\right), L(\overline{u}(x,\alpha))\right],$$

where

$$L\left(\underline{u}(x,\alpha)\right) = \int_0^\infty e^{-sx}\underline{u}(x,\alpha)dx = \lim_{\tau\to\infty}\int_0^\tau e^{-sx}\underline{u}(x,\alpha)dx\,, 0 \le \alpha \le 1$$

$$L\left(\overline{u}(x,\alpha)\right) = \int_0^\infty e^{-sx}\overline{u}(x,\alpha)dx = \lim_{\tau\to\infty}\int_0^\tau e^{-sx}\overline{u}(x,\alpha)dx\,, \quad 0 \le \alpha \le 1$$

Theorem(2.2)[**44**]: Suppose that $u(x)$ *and* $v(x)$ are continuous fuzzy valued functions, and suppose c_1, c_2 are constants, then:

$$L\{c_1\, u(x) + c_2 v(x)\} = c_1\, L\{u(x)\} + c_2\, L\{v(x)\} = c_1 U(s) + c_2 V(s)$$

Lemma(2.1) [44]: Suppose that $u(x)$ be a fuzzy valued function on $[\,0,\infty)$ and $c \in R$, then

$$L\{c\, u(x)\} = c\, L\{u(x)\} = cU(s)$$

Lemma(2.2) [44]: Suppose that $u(x)$ is a continuous fuzzy valued function and $v(x) \ge 0$ is a real valued function such that $(u(x), v(x))e^{-sx}$ is improper fuzzy Riemann-integrable on $[\,0,\infty\,)$ then for fixed $\alpha \in [0,1]$, we have

$$\int_0^\infty (u(x,\alpha)v(x,\alpha))\, e^{-sx}dx$$

$$= \left[\int_0^\infty \underline{u}(x,\alpha)v(x,\alpha)e^{-sx}dx\,, \int_0^\infty \overline{u}(x,\alpha)v(x,\alpha)e^{-sx}dx\right]$$

Theorem(2.3)[**44**]: **(First shifting theorem):** Let $u(x)$ be a continuous fuzzy valued function and $L\{u(x,\alpha)\} = U(s);\ then$:

$L\{e^{ax}u(x)\} = U(s-a),\ s-a > 0.$

Proof:

$$L\{e^{ax}u(x)\} = \int_0^\infty e^{-sx}e^{ax}u(x)dx$$
$$= \int_0^\infty e^{-sx+ax}u(x)dx = \int_0^\infty e^{-(s-a)x}u(x)dx = U(s-a)$$

hence,

$$L\{e^{ax}u(x)\} = U(s-a).$$

Theorem(2.4)[**15**]: Let $u(x)$ be a continuous fuzzy valued function and $L\{u(x)\} = U(s)$. Suppose that c is a constant, then:

$$L\{u(cx)\} = \frac{1}{c}U\left(\frac{s}{c}\right)$$

Proof:

$$L\{u(cx)\} = \int_0^\infty e^{-sx}u(cx)dx$$

$$put\ y =\ cx \Longrightarrow \frac{dy}{dx} = c \Longrightarrow dx = \frac{dy}{c}$$

now,

$$L\{u(cx)\} = \int_0^\infty e^{-s\frac{y}{c}}u(y)\frac{dy}{c} = \frac{1}{c}\int_0^\infty e^{-s\frac{y}{c}}u(y)dy = \frac{1}{c}U\left(\frac{s}{c}\right)$$

hence,

$$L\{u(cx)\} = \frac{1}{c}U\left(\frac{s}{c}\right).$$

2.2 Fuzzy Convolution

In this section we will introduce the concept of fuzzy convolution, and solve fuzzy convolution Volterra integral equation of the second kind.

Definition $(\mathbf{2.2})$ **[42]:** $(\boldsymbol{Jump\ discontinuity})$**:** Suppose that $u(x)$ is a fuzzy-valued function, u is said to have a jump discontinuity at x_0 if the following conditions are satisfied:

- $\lim_{x \to x_0 -} u(x)\, and \ \lim_{x \to x_0 +} u(x) \ exist$
- $u(x_0^-) \neq u(x_0^+)$

Definition $(\mathbf{2.3})$**[42]:** $(\boldsymbol{piecewise\ continuous})$**:** Let $u(x)$ be a fuzzy-valued function,then u is a piecewise continuous on $[\,0, \infty\,)$ if :

- $\lim_{x \to 0^+} u(x) = u(0^+)$
- u is continuous on every finite interval $(0, m)$ except at finite number of points $a_1, a_2, \dots, a_n$ in the interval $(0, m)$ at which u has a jump discontinuity.

Definition $(\mathbf{2.4})$ **[42]:** $(\boldsymbol{exponential\ order})$**:** Suppose that $u(x)$ is a fuzzy-valued function on $[\,0, \infty\,)$, then u has an exponential order a if there exists constants $\varphi > 0$ and a, such that $|u(x)| \leq \varphi e^{ax}$, $x \geq x_0$, for some $x_0 \geq 0$.

Definition $(\mathbf{2.5})$[45]: Fuzzy convolution: Let $p(x)\ and\ q(x)$ are two piece-wise continuous fuzzy-valued functions on $[\,0,\infty\,)$, then the convolution of $p\ and\ q$ denoted by $p * q$ is defined as:

$$(p * q)(x) = \int_0^x p(\tau)q(x-\tau)d\tau\,. \qquad (2.2)$$

Properties of fuzzy convolution:

$(\,i\,)\ \ (p * q) = (q * p) \qquad (commutative\ property)$

$(\,ii\,)\ \lambda(p * q) = (\lambda p) * q = p * (\lambda q), \lambda\ is\ constant$

$(iii) p * (q * h) = (p * q) * h \qquad (associative\ property)$

Proof:

$(i)\ substituting\ n = x - \tau\ in\ the\ equation\ (2.2),$

$we\ get$:

$$(p * q)(x) = \int_0^x q(n)p(x-n)dx = (q * p)(x)$$

$(ii)\ trivial$

$$(iii)\ [p * (q * h)](x) = \int_0^x p(\tau).(q * h)(x-\tau)d\tau$$

$$= \int_0^x p(\tau)\left(\int_0^{x-\tau} q(\nu)h(x-\tau-\nu)d\nu\right)d\tau$$

$$= \int_0^x \left(\int_0^n p(\tau)\,q(n-\tau)d\tau\right)h(x-n)\,dn$$

$$= [(p*q)*h](x)$$

Theorem$(2.5)[45]$: Convolution theorem:

Suppose that $p(x)\ and\ q(x)$ are piece-wise continuous fuzzy-valued functions on $[\,0,\infty)$ of exponential order a with fuzzy Laplace transforms $P(s)\ and\ Q(s)$ respectively, then:

$$L\{(p*q)(x)\} = L\{p(x)\}.L\{q(x)\} = P(s).Q(s), s > a$$

Proof: See [45]

Definition (2.6)[45]: Fuzzy convolution Volterra integral equation of the second kind: The fuzzy convolution Volterra integral equation of the second kind is defined as:

$$u(x) = f(x) + \lambda\int_0^x k(x-t)u(t)dt\,, x \in [0,T], T < \infty \qquad (2.3)$$

where $k(x-t)$is an arbitrary given fuzzy-valued convolution kernel function, and $f(x)$is a continuous fuzzy-valued function.

now, by taking fuzzy Laplace transform on both sides of equation (2.3), we get:

$$\mathrm{L}\{u(x)\} = L\{f(x)\} + L\left\{\lambda \int_0^x k(x-t)u(t)dt\right\}, x \in [0,T], T < \infty$$

then, by using fuzzy convolution and definition of fuzzy Laplace transform, we get:

$$\begin{cases} L\{\underline{u}(x,\alpha)\} = L\left\{\underline{f}(x,\alpha)\right\} + \lambda \underline{L\{k(x,t)\}}\, \underline{L\{u(x,\alpha)\}} \\ L\{\overline{u}(x,\alpha)\} = L\left\{\overline{f}(x,\alpha)\right\} + \lambda \overline{L\{k(x,t)\}}\, \overline{L\{u(x,\alpha)\}} \end{cases} \quad (2.4)$$

For solving the equation (2.4), we will discuss the following cases:

$case(1): if\ k(x,t) > 0\ ,\ then\ we\ get$:

$$L\{\underline{u}(x,\alpha)\} = L\left\{\underline{f}(x,\alpha)\right\} + \lambda\, L\{\underline{k}(x,t)\} L\{\underline{u}(x,\alpha)\}$$

$$L\{\overline{u}(x,\alpha)\} = L\{\overline{f}(x,\alpha)\} + \lambda L\{\overline{k}(x,t)\} L\{\overline{u}(x,\alpha)\}$$

we obtain explicit formula:

$$L\{\underline{u}(x,\alpha)\} = \frac{L\left\{\underline{f}(x,\alpha)\right\}}{1 - \lambda\, L\{\underline{k}(x,t)\}}$$

$$L\{\overline{u}(x,\alpha)\} = \frac{L\{\overline{f}(x,\alpha)\}}{1 - \lambda L\{\overline{k}(x,t)\}}$$

take the inverse of fuzzy Laplace transform, we get the solution:

$$\underline{u}(x,\alpha) = L^{-1}\left[\frac{\underline{F}(s)}{1 - \lambda K(s)}\right]$$

$$\overline{u}(x,\alpha) = L^{-1}\left[\frac{\overline{F}(s)}{1-\lambda K(s)}\right]$$

where F(s) and K(s) are fuzzy Laplace transformations f(x)and k(x, t) respectively.

$case(2): if\ k(x,t) < 0,\ then\ we\ get$:

$$L\{\underline{u}(x,\alpha)\} = L\left\{\underline{f}(x,\alpha)\right\} + \lambda\, L\{k(x,t)\}L\{\,\overline{u}(x,\alpha)\}$$

$$L\{\overline{u}(x,\alpha)\} = L\{\,\overline{f}(x,\alpha)\} + \lambda L\{k(x,t)\}L\{\,\underline{u}(x,\alpha)\}$$

we obtain explicit formula:

$$L\{\,\underline{u}(x,\alpha)\} = \frac{L\left\{\underline{f}(x,\alpha)\right\} - \lambda L\{\,k(x,t)\}L\{\,\overline{f}(x,\alpha)\}}{1-\lambda^2 L\{\,k(x,t)\}\,L\{\,k(x,t)\}}$$

$$L\{\,\overline{u}(x,\alpha)\} = \frac{L\{\,\overline{f}(x,\alpha)\} - \lambda L\{\underline{k}(x,t)\}L\left\{\underline{f}(x,\alpha)\right\}}{1-\lambda^2 L\{\,k(x,t)\}\,L\{\,k(x,t)\}}$$

take the inverse of fuzzy Laplace transform, we get the solution:

$$\underline{u}(x,\alpha)\ = L^{-1}\left[\frac{\underline{F}(s) - \lambda K(s)\overline{F}(s)}{1-[\lambda\, K(s)]^2}\right]$$

$$\overline{u}(x,\alpha) = L^{-1}\left[\frac{\overline{F}(s) - \lambda K(s)\underline{F}(s)}{1-[\lambda\, K(s)]^2}\right]$$

where F(s) and K(s) are fuzzy Laplace transformations f(x)and k(x, t) respectively.

Example (2.1): consider the following fuzzy Volterra integral equation

$$u(x) = (\alpha + 1,3 - \alpha) + \int_0^x (x - t)\, u(t) dt, x \in [0, T], T < \infty \qquad (2.5)$$

by taking fuzzy Laplace transform on both sides of equation(2.5), we get:

$$L\{u(x)\} = L\{(\alpha + 1,3 - \alpha)\} + L\{x\}. L\{u(t)\}$$

$i.e$

$$L\{\underline{u}(x, \alpha)\} = L\{(\alpha + 1)\ \} + L\{x\}. L\{\underline{u}(x, \alpha)\}, 0 \leq \alpha \leq 1$$

$$L\{\overline{u}(x, \alpha)\} = L\{(3 - \alpha)\} + L\{x\}. L\{\overline{u}(x, \alpha)\}, 0 \leq \alpha \leq 1$$

hence, we get:

$$L\{\underline{u}(x, \alpha)\} = (\alpha + 1)\frac{1}{s} + \frac{1}{s^2} L\{\underline{u}(x, \alpha)\}, 0 \leq \alpha \leq 1$$

$$L\{\overline{u}(x, \alpha)\} = (3 - \alpha)\frac{1}{s} + \frac{1}{s^2} L\{\overline{u}(x, \alpha)\}, 0 \leq \alpha \leq 1$$

Finally, by taking the inverse of fuzzy Laplace transform on both sides of above equations, we get:

$$\begin{cases} \underline{u}(x, \alpha) = (\alpha + 1) \cosh(x) \\ \overline{u}(x, \alpha) = (3 - \alpha) \cosh(x) \end{cases} \qquad 0 \leq \alpha \leq 1$$

2.3 Fuzzy Homotopy Analysis Method

In this section, we find the approximate analytical solutions of fuzzy Volterra integral equations of the second kind by using fuzzy homotopy analysis method. The main advantage of this method is that it can be used directly without using assumptions or transformations. The (FHAM) solutions depend on an auxiliary parameter which provides a suitable way of controlling the convergence region of series solutions. we give an example reveal the efficiency of this method**.**

Consider the following differential equation:

$$\mathcal{M}\big(u(x,\alpha)\big) = 0, \qquad (2.6)$$

where $\mathcal{M}$ is a nonlinear operator, x denotes independent variable and $u(x,\alpha)$is the unknown function.

Definition $(\mathbf{2.7})$ [31]: Homotopy operator: We define the homotopy operator $\mathcal{H}$ as:

$$\mathcal{H}(\Phi,q) \equiv (1-q)\mathcal{L}\big(\Phi(x,q,\alpha) - u_0(x,\alpha)\big) - q\ s\ S(x,\alpha)\ \mathcal{M}\big(\Phi(x,q,\alpha)\big)$$
(2.7)

where $q \in [0,1]$ is the embedding parameter, $s \neq 0$ is a non- zero auxiliary parameter (convergence control parameter), $S(x,\alpha) \neq 0$ is an auxiliary function, $u_0(x,\alpha)$ denotes the initial approximation of $u(x,\alpha)$ in the equation (2.6)$\Phi(x,q,\alpha)$ is an unknown function, and $\mathcal{L}$ is an auxiliary linear operator with property $\mathcal{L}(0) = 0$.

In equation (2.7) if we put $\mathcal{H}(\Phi,q,\alpha) = 0$, we will construct the so-called $zero$ order deformation equation

$$(1-q)\mathcal{L}\big(\Phi(x,q,\alpha) - u_0(x,\alpha)\big) = q\ s\ S(x,\alpha)\ \mathcal{M}\big(\Phi(x,q,\alpha)\big) \qquad (2.8)$$

obviously, for $q = 0$ we get

$$\mathcal{L}\big(\Phi(x,0,\alpha) - u_0(x,\alpha)\big) = 0 \Longrightarrow \Phi(x,0,\alpha) = u_0(x,\alpha)$$

whereas, for $q = 1$,we get:

$\mathcal{M}\big(\Phi(x,1,\alpha)\big) = 0 \Longrightarrow \Phi(x,1,\alpha) = u(x,\alpha)$, where $u(x)$ is

the solution of the equation (2.6)

thus, as q increases from $0\ to\ 1$, the solution $\Phi(x,q,\alpha)$ changes continuously from the initial approximation $u_0(x,\alpha)$ to the exact solution $u(x,\alpha)$.

by expanding $\Phi(x,q,\alpha)$ into the Maclaurin series with respect to q,

we have:

$$\Phi(x,q,\alpha) = u_0(x,\alpha) + \sum_{n=1}^{\infty} u_n(x,\alpha)q^n \tag{2.9}$$

where

$$u_n(x,\alpha) = D_n(\Phi) = \left[\frac{1}{n!}\,\frac{\partial^n \mathcal{M}\big(\Phi(x,q,\alpha)\big)}{\partial q^n}\right]_{q=0}, n = 1,2,\dots \tag{2.10}$$

if the series (2.9) convergent for $q = 1$, then we have

$$u(x,\alpha) = u_0(x,\alpha) + \sum_{n=1}^{\infty} u_n(x,\alpha) \tag{2.11}$$

In $[\mathbf{16}]$ by using equation (2.9), we can write equation (2.8) as:

$$(1-q)\mathcal{L}\big(\Phi(x,q,\alpha)-u_0(x,\alpha)\big) = (1-q)\mathcal{L}\left[\sum_{n=1}^{\infty} u_n(x,\alpha)q^n\right]$$

$$= q\,s\,S(x)\mathcal{M}\big(\Phi(x,q,\alpha)\big) \qquad (2.12)$$

this implies

$$\mathcal{L}\left[\sum_{n=1}^{\infty} u_n(x,\alpha)q^n\right] - q\mathcal{L}\left[\sum_{n=1}^{\infty} u_n(x,\alpha)q^n\right] = q\,s\,S(x)\mathcal{M}\big(\Phi(x,q,\alpha)\big)$$

$$(2.13)$$

by differentiating the both sides of equation $(2.13) n\ times$ with respect to q, we get:

$$n!\,\mathcal{L}[u_n(x,\alpha)-u_{n-1}(x,\alpha)] = s\,S(x)n\left[\frac{\partial^{n-1}\mathcal{M}(\Phi(x,q,\alpha))}{\partial q^{n-1}}\right]_{q=0}$$

hence, the nth-order deformation equation $(n>0)$ is:

$$\mathcal{L}[u_n(x,\alpha)-\chi_n u_{n-1}(x,\alpha)] = s\,S(x)\,\mathrm{T}_n(\vec{u}_{n-1}(x,\alpha)) \qquad (2.14)$$

$where\quad \vec{u}_{n-1} = \{u_0(x,\alpha), u_1(x,\alpha), \dots, u_{n-1}(x,\alpha)\},$

$$\chi_n = \begin{cases} 0, & n \le 1 \\ 1, & n > 1 \end{cases} \qquad (2.15)$$

and

$$\mathrm{T}_n(\vec{u}_{n-1}(x,\alpha)) = \left[\frac{1}{(n-1)!}\frac{\partial^{n-1}}{\partial q^{n-1}}\mathcal{M}\big(\Phi(x,q,\alpha)\big)\right]_{q=0} \qquad (2.16)$$

To ensure the convergence of the series, we must concentrate that $u_0(x,\alpha)$ is the initial approximation, $s \neq 0$ is a non-zero auxiliary parameter, $\mathcal{L}$ is an auxiliary linear operator, q is the embedding parameter, and $S(x,\alpha)$ is an auxiliary function.

In this part, we will rewrite the fuzzy Volterra integral equations of the second kind, and then solve them by using homotopy analysis method.

we know that the parametric form of fuzzy Volterra integral equation of second kind is:

$$\begin{cases} \underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_{a}^{x} k(x,t)\underline{u}(t,\alpha)dt \\ \overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_{a}^{x} k(x,t)\overline{u}(t,\alpha)dt \end{cases} \tag{2.17}$$

now, assume that $k(x,t) \geq 0 \, , a \leq s \leq c, and\ k(x,t) < 0, c \leq s \leq x$

with above assumptions, the equation (2.17) becomes:

$$\begin{cases} \underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_{a}^{c} k(x,t)\underline{u}(t,r)dt + \lambda \int_{c}^{x} k(x,t)\overline{u}(t,\alpha)dt \\ \overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_{a}^{c} k(x,t)\overline{u}(t,\alpha)dt + \lambda \int_{c}^{x} k(x,t)\underline{u}(t,\alpha)dt \end{cases} \tag{2.18}$$

we note that (2.18) is a linear Fredholm-Volterra integral equations in crisp case for each $0 \leq \alpha \leq 1$.

To solve the system (2.18) by means of homotopy analysis method, we choose the linear operator [**35**]:

$$\mathcal{L}[\Phi(x,q,\alpha)] = \Phi(x,q,\alpha) \tag{2.19}$$

we now define a nonlinear operator $\mathcal{M}\big(\Phi(x,q,r)\big)$ as:

$$\underline{\mathcal{M}}\left(\underline{\Phi}(x,q,\alpha)\right) = \underline{\Phi}(x,q,\alpha) - \underline{f}(x,\alpha) - \lambda \int_a^c k(x,t)\underline{\Phi}(t,q,\alpha)dt$$

$$-\lambda \int_c^x k(x,t)\overline{\Phi}(t,q,\alpha)dt,$$

and

$$\overline{\mathcal{M}}\left(\overline{\Phi}(x,q,\alpha)\right) = \overline{\Phi}(x,q,\alpha) - \overline{f}(x,\alpha) - \lambda \int_a^c k(x,t)\overline{\Phi}(t,q,\alpha)dt$$

$$-\lambda \int_c^x k(x,t)\underline{\Phi}(t,q,\alpha)dt. \tag{2.20}$$

using the two equations (2.8) and (2.20), we construct the $zeroth\text{-}order$ deformation equation:

$$\begin{cases} (1-q)\mathcal{L}\left[\underline{\Phi}(x,q,\alpha) - \underline{u}_0(x,\alpha)\right] = q\ s\ \underline{S}(x,\alpha)\underline{\mathcal{M}}\left(\underline{\Phi}(x,q,\alpha)\right) \\ (1-q)\mathcal{L}\left[\overline{\Phi}(x,q,\alpha) - \overline{u}_0(x,\alpha)\right] = q\ s\ \overline{S}(x,\alpha)\overline{\mathcal{M}}\left(\overline{\Phi}(x,q,\alpha)\right) \end{cases} \tag{2.21}$$

Substituting equation (2.19) and (2.20) $into$ (2.21), we obtain:

$$(1-q)[\,\underline{\Phi}(x,q,\alpha)-\underline{u}_0(x,\alpha)] = q\ s\ \underline{S}(x,\alpha)[\,\underline{\Phi}(x,q,\alpha)-\underline{f}(x,\alpha) -\lambda\int_a^c k(x,t)\underline{\Phi}(t,q,\alpha)dt -\lambda\int_c^x k(x,t)\overline{\Phi}(t,q,\alpha)dt],$$

and

$$(1-q)[\overline{\Phi}(x,q,\alpha)-\overline{u}_0(x,\alpha)] = q\ s\ \overline{S}(x,\alpha)\overline{\Phi}(x,q,\alpha)-\overline{f}(x,\alpha) -\lambda\int_a^c k(x,t)\overline{\Phi}(t,q,\alpha)dt -\lambda\int_c^x k(x,t)\underline{\Phi}(t,q,\alpha)dt \qquad (2.22)$$

take $S(x,\alpha)=1$,when $q=0$, the zero-order deformation (2.22) becomes:

$$\begin{cases}\underline{\Phi}(x,0,\alpha)=\underline{u}_0(x,\alpha)\\ \overline{\Phi}(x,0,\alpha)=\overline{u}_0(x,\alpha)\end{cases} \qquad (2.23)$$

when $q=1$, the $zeroth\ order$ deformation (2.22) becomes:

$$\begin{cases}\underline{\Phi}(x,1,\alpha)=\underline{f}(x,\alpha)+\lambda\int_a^c k(x,t)\underline{\Phi}(t,1,\alpha)dt+\lambda\int_c^x k(x,t)\overline{\Phi}(t,1,\alpha)dt\\ \overline{\Phi}(x,1,\alpha)=\overline{f}(x,\alpha)+\lambda\int_a^c k(x,t)\overline{\Phi}(t,1,\alpha)dt+\lambda\int_c^x k(x,t)\underline{\Phi}(t,1,\alpha)dt\end{cases}$$

$$(2.24)$$

notice that this equation is the same as equation (2.18). Thus, as q increases from 0 to 1, the analytical solution $\left(\underline{\Phi}(x,q,\alpha),\overline{\Phi}(x,q,\alpha)\right)$ changes from the initial approximation $\left(\underline{u}_0(x,\alpha),\overline{u}_0(x,\alpha)\right)$ to the exact solution $\left(\underline{u}(x,\alpha),\overline{u}(x,\alpha)\right)$.

Expanding $\underline{\Phi}(x,q,\alpha), and\ \overline{\Phi}(x,q,\alpha)$ into the Maclaurin series with respect to q, we have:

$$\begin{cases}\underline{\Phi}(x,q,\alpha) = \underline{u}_0(x,\alpha) + \sum_{n=1}^{\infty} \underline{u}_n(x,\alpha)q^n \\ \overline{\Phi}(x,q,\alpha) = \overline{u}_0(x,\alpha) + \sum_{n=1}^{\infty} \overline{u}_n(x,\alpha)q^n\end{cases} \tag{2.25}$$

where

$$\underline{u}_n(x,\alpha) = \left[\frac{1}{n!}\ \frac{\partial^n \underline{\Phi}(x,q,\alpha)}{\partial q^n}\right]_{q=0}$$

$$\overline{u}_n(x,\alpha) = \left[\frac{1}{n!}\ \frac{\partial^n \overline{\Phi}(x,q,\alpha)}{\partial q^n}\right]_{q=0} \tag{2.26}$$

Differentiating equation (2.22) n times with respect to q, we obtain [**26**]:

$$\frac{\partial^n \underline{\Phi}(x,q,\alpha)}{\partial q^n} - \frac{\partial^{n-1} \underline{\Phi}(x,q,\alpha)}{\partial q^{n-1}} = s[\frac{\partial^{n-1} \underline{\Phi}(x,q,\alpha)}{\partial q^{n-1}} - \underline{f}(x,\alpha) -\lambda \int_a^c k(x,t)\frac{\partial^{n-1} \underline{\Phi}(t,q,\alpha)}{\partial q^{n-1}}dt -\lambda \int_c^x k(x,t)\frac{\partial^{n-1} \overline{\Phi}(t,q,\alpha)}{\partial q^{n-1}}dt],$$

and

$$\frac{\partial^n \overline{\Phi}(x,q,\alpha)}{\partial q^n} - \frac{\partial^{n-1} \overline{\Phi}(x,q,\alpha)}{\partial q^{n-1}} = s[\frac{\partial^{n-1} \overline{\Phi}(x,q,\alpha)}{\partial q^{n-1}} - \overline{f}(x,\alpha) -\lambda \int_a^c k(x,t)\frac{\partial^{n-1} \overline{\Phi}(t,q,\alpha)}{\partial q^{n-1}}dt -\lambda \int_c^x k(x,t)\frac{\partial^{n-1} \underline{\Phi}(t,q,\alpha)}{\partial q^{n-1}}dt]. \tag{2.27}$$

now, dividing the equation(2.27) by $n!$, and setting $q = 0$, we get the nth-order deformation equation:

$$\underline{u}_n(x,\alpha) = \mu_n \, \underline{u}_{n-1}(x,\alpha) + s[\, \underline{u}_{n-1}(x,\alpha) - \varepsilon_n \underline{f}(x,\alpha)$$
$$- \lambda \int_a^c k(x,t)\underline{u}_{n-1}(t,q,\alpha)dt - \lambda \int_c^x k(x,t)\overline{u}_{n-1}(t,q,\alpha)dt]$$

(2.27)

$$\overline{u}_n(x,\alpha) = \mu_n \, \overline{u}_{n-1}(x,\alpha) + s[\, \overline{u}_{n-1}(x,\alpha) - \varepsilon_n \overline{f}(x,\alpha)$$
$$- \lambda \int_a^c k(x,t)\overline{u}_{n-1}(t,q,\alpha)dt - \lambda \int_c^x k(x,t)\underline{u}_{n-1}(t,q,\alpha)dt]$$

where $n \geq 1,\ and$

$$\mu_n = \begin{cases} 0, & n = 1 \\ 1, & n \neq 1 \end{cases},$$

$$\varepsilon_n = \begin{cases} 0, & n \neq 1 \\ 1, & n = 1 \end{cases},$$

If we take $\underline{u}_0(x,\alpha) = \overline{u}_0(x,\alpha) = \overline{0}$, for $n \geq 2$, we have:

$$\underline{u}_n(x,\alpha) = (s+1)\underline{u}_{n-1}(x,\alpha) - s\,\lambda(\int_a^c k(x,t)\underline{u}_{n-1}(t,q,\alpha)dt$$
$$+ \int_c^x k(x,t)\overline{u}_{n-1}(t,q,\alpha)dt\,),$$

$$\overline{u}_n(x,\alpha) = (s+1)\overline{u}_{n-1}(x,\alpha) - s\lambda(\int_a^c k(x,t)\overline{u}_{n-1}(t,q,\alpha)dt$$

$$+ \int_c^x k(x,t)\underline{u}_{n-1}(t,q,\alpha)dt\,). \qquad (2.29)$$

hence, the solution of the system (2.18) in series form is:

$$\underline{u}(x,\alpha) = \lim_{q\to 1} \underline{\Phi}\,(x,q,\alpha) = \sum_{n=1}^{\infty} \underline{u}_n\,(x,\alpha),$$
$$\overline{u}(x,\alpha) = \lim_{q\to 1} \overline{\Phi}\,(x,q,\alpha) = \sum_{n=1}^{\infty} \overline{u}_n\,(x,\alpha). \tag{2.30}$$

We denote the jth −order approximation to the solutions $\underline{u}(x,\alpha), and\ \overline{u}(x,\alpha)$ respectively with:

$$\underline{u}_j(x,\alpha) = \sum_{n=1}^{j} \underline{u}_n\,(x,\alpha),$$

$$\overline{u}_j(x,\alpha) = \sum_{n=1}^{j} \overline{u}_n\,(x,\alpha).$$

Example $(\mathbf{2.2})$:

Consider the fuzzy Volterra integral equation

$$u(x) = (\alpha+1,3-\alpha) + \int_0^x (x-t)\,u(t,\alpha)dt, x \in [0,T], T < \infty$$

Solution:

In this example, $k(x,t) \geq 0$ for each $0 \leq t \leq x,$ in this case $c = x,$

$a = 0, b = 1.$ By equation (2.27), the first terms of homotopy analysis method series are:

$\underline{u}_0(x,\alpha) = \underline{0},$

$\underline{u}_1(x,\alpha) = -s\underline{f}(x,\alpha) = -s(\alpha+1)$

$$\underline{u}_2(x,\alpha) = (1+s)\underline{u}_1(x,\alpha) - s\int_0^x (x-t)\,\underline{u}_1(t,\alpha)dt$$

$$= -s(\alpha+1)\left[1+s\left(1-\frac{x^2}{2}\right)\right]$$

$$\underline{u}_3(x,\alpha) = \ (1+s)\ \underline{u}_2(x,\alpha) - s\ \int_0^x (x-t)\ \underline{u}_2(t,\alpha)dt$$

$$= \ -s(\alpha+1)\left[\ 1+s(2-x^2)+s^2\left(1-x^2+\frac{x^4}{24}\right)\right]$$

and

$$\overline{u}_0(x,\alpha) = \overline{0}$$

$$\overline{u}_1(x,\alpha) = \ -s\overline{f}(x,\alpha) = -s(3-\alpha)$$

$$\overline{u}_2(x,\alpha) = (1+s)\overline{u}_1(x,\alpha) - s\int_0^x (x-t)\,\overline{u}_1(t,\alpha)dt$$

$$= -s(3-\alpha)\left[1+s\left(1-\frac{x^2}{2}\right)\right]$$

$$\overline{u}_3(x,\alpha) = \ (1+s)\ \overline{u}_2(x,\alpha) - s\ \int_0^x (x-t)\overline{u}_2\ (t,\alpha)dt$$

$$= \ -s(3-\alpha)\left[\ 1+s(2-x^2)+s^2\left(1-x^2+\frac{x^4}{24}\right)\right]$$

then we approximate $\underline{u}(x,\alpha)$ with

$$\underline{u}_3(x,\alpha) = \sum_{n=1}^{3} \underline{u}_n\ (x,\alpha)$$

$$= -s\ (\alpha+1)\left[3+s\left(3-\frac{3}{2}x^2\right)+s^2\left(1-x^2+\frac{x^4}{24}\right)\right]$$

and $\overline{u}(x,\alpha)$ with

$$\overline{u}(x,\alpha)=\sum_{n=1}^{3}\overline{u}_n\,(x,\alpha)$$

$$=-s\,(3-\alpha)\left[3+s\left(3-\frac{3}{2}x^2\right)+s^2\left(1-x^2+\frac{x^4}{24}\right)\right]$$

The exact solution of system is given by:

$$\underline{u}(x,\alpha)=\lim_{q\to 1}\underline{\Phi}(x,q,\alpha)=\sum_{n=1}^{\infty}\underline{u}_n(x,\alpha),$$

and

$$\overline{u}(x,\alpha)=\lim_{q\to 1}\overline{\Phi}(x,q,\alpha)=\sum_{n=1}^{\infty}\overline{u}_n(x,\alpha).$$

hence, the solution is:

$$\underline{u}(x,\alpha)=-s\,(\alpha+1)\left[3+s\left(3-\frac{3}{2}x^2\right)+s^2\left(1-x^2+\frac{x^4}{24}\right)+\cdots\right],$$

$$\overline{u}(x,\alpha)=-s\,(3-\alpha)\left[3+s\left(3-\frac{3}{2}x^2\right)+s^2\left(1-x^2+\frac{x^4}{24}\right)+\cdots\right]$$

note that the solution series contains the auxiliary parameter s, which provides a convenient way to controlling the convergence region of series solution. we choose $s=-1$, we have:

$$\underline{u}(x,\alpha)=(\alpha+1)\left[1+\frac{x^2}{2}+\frac{x^4}{24}+\cdots\right]=(\alpha+1)\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

$$\overline{u}(x,\alpha)=\ (3-\alpha)\left[1+\frac{x^2}{2}+\frac{x^4}{24}+\cdots\right]=(3-\alpha)\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}$$

as $n \to \infty$, the solution series converges to:

$$\begin{cases} \underline{u}(x,\alpha) = (\alpha+1)\cosh(x) \\ \overline{u}(x,\alpha) = \ (3-\alpha)\cosh(x) \end{cases} \qquad 0 \le \alpha \le 1$$

2.4 Adomian Decomposition Method (ADM)

The Adomian decomposition method has been applied by scientists and engeneers since the beginning of the 1980(s)[15]. this method gives the solution as infinite series usually converges to the closed form solution.

The Adomian decomposition method is a special case of homotopy analysis method, if we take $s = -1$ in a homotopy frame then it's converted to the Adomian decomposition method. Consider the fuzzy Volterra integral equation:

$$\begin{cases} \underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^x \underline{k(x,t)\big(u(t,\alpha)\big)}\, dt \\ \overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_a^x \overline{k(x,t)\big(u(t,\alpha)\big)}\, dt \end{cases} \tag{2.31}$$

where $a \le t \le x, x \in [a,b], 0 \le \alpha \le 1, and$

$$\underline{k(x,t)u(x,\alpha)} = \begin{cases} k(x,t)\underline{u}(x,\alpha), k(x,t) \ge 0, \\ k(x,t)\overline{u}(x,\alpha), k(x,t) < 0. \end{cases}$$

$$\overline{k(x,t)u(x,\alpha)} = \begin{cases} k(x,t)\overline{u}(x,\alpha), k(x,t) \ge 0, \\ k(x,t)\underline{u}(x,\alpha), k(x,t) < 0. \end{cases}$$

for each $0 \le \alpha \le 1$, and $a \le x \le b$.

In order to solve equation(2.31) by the Adomian decomposition method, we express the solution for the unknown functions

[$\underline{u}(x,\alpha), \overline{u}(x,\alpha)$] in series form[15]:

$$\begin{cases} \underline{u}(x,\alpha) = \sum_{r=0}^{\infty} \underline{u}_r(x,\alpha) \\ \overline{u}(x,\alpha) = \sum_{r=0}^{\infty} \overline{u}_r(x,\alpha) \end{cases} \tag{2.32}$$

now, setting (2.32) $into$ (2.31) yields:

$$\begin{cases} \sum_{n=0}^{\infty} \underline{u}_n(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^x k(x,t)\left(\sum_{n=0}^{\infty} \underline{u}_n(t,\alpha)\right) dt \\ \sum_{n=0}^{\infty} \overline{u}_n(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_a^x k(x,t)\left(\sum_{n=0}^{\infty} \overline{u}_n(t,\alpha)\right) dt \end{cases} \tag{2.33}$$

or

$$\begin{cases} \underline{u}_0(x,\alpha) + \cdots = \underline{f}(x,\alpha) + \lambda \int_a^x k(x,t)\big(\underline{u}_0(t,\alpha) + \underline{u}_1(t,\alpha) + \cdots\big)dt \\ \overline{u}_0(x,\alpha) + \cdots = \overline{f}(x,\alpha) + \lambda \int_a^x k(x,t)(\overline{u}_0(t,\alpha) + \overline{u}_1(t,\alpha) + \cdots)dt \end{cases}$$

(2.34)

The $zeroth\ component$ can be identified by all the terms not included under the integral sign, that is:

$$\begin{cases} \underline{u}_0(x,\alpha) = \underline{f}(x,\alpha) \\ \overline{u}_0(x,\alpha) = \overline{f}(x,\alpha) \end{cases} \tag{2.35}$$

whereas the $jth - components, j \geq 1$ are given by the recurrence relation:

$$\underline{u}_{n+1}(x,\alpha) = \lambda \int_a^x k(x,t)\big(\,\underline{u}_n(t,\alpha)\,\big)dt,$$

$$(2.36)$$

$$\overline{u}_{n+1}(x,\alpha) = \lambda \int_a^x k(x,t)(\,\overline{u}_n(t,\alpha)\,)dt, \quad n \geq 0$$

we approximate $u(x,\alpha) = \big[\,\underline{u}(x,\alpha),\ \overline{u}(x,\alpha)\,\big]$ by [45]:

$$\underline{\psi_n} = \sum_{r=0}^{n-1} \underline{u}_r(x,\alpha) \quad and \quad \overline{\psi}_n = \sum_{r=0}^{n-1} \overline{u}_r(x,\alpha)$$

where

$$\lim_{n\to\infty} \underline{\psi_n} = \underline{u}(x,\alpha)$$

and

$$\lim_{n\to\infty} \overline{\psi}_n = \overline{u}(x,\alpha)$$

Example $(\mathbf{2.3})$:

Consider the fuzzy Volterra integral equation:

$$u(x) = (\alpha + 1, 3 - \alpha) + \int_0^x (x-t)\,u(t)dt, x \in [0,T], T < \infty$$

Solution:

In this example $k(x,t) \geq 0, for\ each\ 0 \leq t \leq x$.

the first terms of Adomain decomposition series are:

$$\underline{u}_0(x,\alpha) = (\alpha+1)$$

$$\underline{u}_1(x,\alpha) = \int_0^x (x-t)\,\underline{u}_0(t,\alpha)dt = \int_0^x (x-t)\,(\alpha+1)dt = (\alpha+1)\left(\frac{x^2}{2}\right)$$

$$\underline{u}_2(x,\alpha) = \int_0^x (x-t)\,\underline{u}_1(t,\alpha)dt = \int_0^x (x-t)\left(\frac{t^2}{2}\right)(\alpha+1)dt$$
$$= (\alpha+1)\left(\frac{x^4}{24}\right)$$

$$\underline{u}_3(x,\alpha) = \int_0^x (x-t)\,\underline{u}_2(t,\alpha)dt = \int_0^x (x-t)\left(\frac{t^4}{24}\right)(\alpha+1)dt$$
$$= (\alpha+1)\left(\frac{x^6}{720}\right)$$

and

$$\overline{u}_0(x,\alpha) = (3-\alpha)$$

$$\overline{u}_1(x,\alpha) = \int_0^x (x-t)\,\overline{u}_0(t,\alpha)dt = \int_0^x (x-t)\,(3-\alpha)dt = (3-\alpha)\left(\frac{x^2}{2}\right)$$

$$\overline{u}_2(x,\alpha) = \int_0^x (x-t)\,\overline{u}_1(t,\alpha)dt = \int_0^x (x-t)\left(\frac{t^2}{2}\right)(3-\alpha)dt$$
$$= (3-\alpha)\left(\frac{x^4}{24}\right)$$

$$\overline{u}_3(x,\alpha) = \int_0^x (x-t)\,\overline{u}_2(t,\alpha)dt = \int_0^x (x-t)\left(\frac{t^4}{24}\right)(3-\alpha)dt$$
$$= (3-\alpha)\left(\frac{x^6}{720}\right)$$

then, we approximate $u(x,\alpha) = \left[\underline{u}(x,\alpha),\ \overline{u}(x,\alpha)\right]$ by:

$$\underline{\psi}_4 = \sum_{r=0}^{3} \underline{u}_r(x,\alpha)$$
$$= (\alpha+2)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}\right)$$

and

$$\overline{\psi}_4 = \sum_{r=0}^{3} \overline{u}_r(x,\alpha)$$
$$= (3-\alpha)\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}\right)$$

$$\underline{u}(x,\alpha) = \lim_{n\to\infty} \underline{\psi}_n = (\alpha+1)\lim_{n\to\infty}\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)$$
$$= (\alpha+1)\cosh(x),$$
$$\overline{u}(x,\alpha) = \lim_{n\to\infty} \overline{\psi}_n = (3-\alpha)\lim_{n\to\infty}\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)$$
$$= (3-\alpha)\cosh(x).$$

2.5 Fuzzy Differential Transformation Method

In this section FDTM is applied to fuzzy Volterra integral equation of the second kind, an example is shown to illustrate the superiority of this method.

As special case of definition (1.23) if $u(x)$ is a fuzzy-valued function, we get the following result.

Definition$(2.8)[11]$: (Strongly generalized differentiable of the $\boldsymbol{n-th}$ order at $\boldsymbol{x_0}$)

Let $u:(a,b) \to E$ and $x_0 \in (a,b)$. we say that u is strongly generalized differential of the $nth\ order$ at x_0. If there exists an element $u^{(k)}(x_0) \in E, \forall k = 1,2,\dots,n,$ such that:

(i) $\forall\, h > 0$ Sufficiently small, $\exists u^{(k-1)}(x_0+h) - u^{(k-1)}(x_0)$, $\exists u^{(k-1)}(x_0) - u^{(k-1)}(x_0-h)$, and the limits (in the metric d)

$$\lim_{h\to 0}\frac{u^{(k-1)}(x_0+h) - u^{(k-1)}(x_0)}{h} = \lim_{h\to 0}\frac{u^{(k-1)}(x_0) - u^{(k-1)}(x_0-h)}{h}$$
$$= u^{(k)}(x_0)$$

(ii) $\forall\, h > 0$ Sufficiently small, $\exists u^{(k-1)}(x_0) - u^{(k-1)}(x_0+h)$,

$\exists u^{(k-1)}(x_0-h) - u^{(k-1)}(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}\frac{u^{(k-1)}(x_0) - u^{(k-1)}(x_0+h)}{h} = \lim_{h\to 0}\frac{u^{(k-1)}(x_0-h) - u^{(k-1)}(x_0)}{h}$$
$$= u^{(k-1)}(x_0),$$

(iii) $\forall\, h > 0$ Sufficiently small, $\exists u^{(k-1)}(x_0)(x_0+h) - u^{(k-1)}(x_0)$, $\exists u^{(k-1)}(x_0)(x_0-h) - u^{(k-1)}(x_0)$ and the limits (in the metric d)

$$\lim_{h\to 0}\frac{u^{(k-1)}(x_0)-u^{(k-1)}(x_0+h)}{-h}=\lim_{h\to 0}\frac{u^{(k-1)}(x_0)-u^{(k-1)}(x_0-h)}{-h}$$

$$=u^{(k)}(x_0),$$

(iv) $\forall\, h>0$ Sufficiently small, $\exists u^{(k-1)}(x_0)-u^{(k-1)}(x_0+h)$,

$\exists u^{(k-1)}(x_0)-u^{(k-1)}(x_0-h)$, and the limits (in the metric d)

$$\lim_{h\to\to 0}\frac{u^{(k-1)}(x_0)-u^{(k-1)}(x_0+h)}{-h}=\lim_{h\to 0}\frac{u^{(k-1)}(x_0)-u^{(k-1)}(x_0-h)}{h}$$

$$=u^{(k)}(x_0).$$

Theorem$(2.6)[11]$:

Let $u\colon R\to E$ be a fuzzy-valued function denoted by:

$u(x)=\left(\underline{u}(x,\alpha),\overline{u}(x,\alpha)\right), for\ each\ \alpha\in[0,1], then$:

(1) If $u(x)$ is $(i)-$ differentiable, then $\underline{u}(x,\alpha)\ and\ \overline{u}(x,\alpha)$ are differentiable functions, and $u'(x)=\left(\underline{u'}(x,\alpha),\overline{u}'(x,\alpha)\right)$.

(2) If $u(x)$ is $(ii)-$ differentiable, then $\underline{u}(x,\alpha)\ and\ \overline{u}(x,\alpha)$ are differentiable functions, and $u'(x)=\left(\ \overline{u}'(x,\alpha)\,,\underline{u'}(x,\alpha)\ \right)$.

Definition (2.9) $[37]$:

Let $u(x,\alpha)$ be strongly generalized differentiable of order n in time domain T, then if $u(x,\alpha)$ is differentiable in first form (i)

$$\underline{\varphi}\,(x,n,\alpha)=\frac{d^n\left(\underline{u}\,(x,\alpha)\right)}{d\,x^n}\,,\overline{\varphi}\,(x,n,\alpha)=\frac{d^n\left(\overline{u}\,(x,\alpha)\right)}{d\,x^n}\,,\quad \forall x\in T$$

then

$$U_i(n,\alpha) = \begin{cases} \underline{U}_i(n,\alpha) = \underline{\varphi}\,(x_i,n,\alpha) = \left.\dfrac{d^n\left(\underline{u}\,(x,\alpha)\right)}{d\,x^n}\right|_{x=x_i} \\ \overline{U}_i(n,\alpha) = \overline{\varphi}(x_i,n,\alpha) = \left.\dfrac{d^n\left(\overline{u}\,(x,\alpha)\right)}{d\,x^n}\right|_{x=x_i} \end{cases} ,\forall n \in N$$

If $u(x,\alpha)$ is differentiable in second form (ii)

$$\underline{\varphi}\,(x,n,\alpha) = \frac{d^n\left(\overline{u}\,(x,\alpha)\right)}{d\,x^n}\,, \overline{\varphi}\,(x,n,\alpha) = \frac{d^n\left(\underline{u}\,(x,\alpha)\right)}{d\,x^n}\,, \quad \forall x \in T$$

then

$$U_i(n,\alpha) = \begin{cases} \underline{U}_i(n,\alpha) = \overline{\varphi}\,(x_i,n,\alpha) = \left.\dfrac{d^n\left(\underline{u}(x,\alpha)\right)}{d\,x^n}\right|_{x=x_i} \\ \overline{U}_i(n,\alpha) = \underline{\varphi}\,(x_i,n,\alpha) = \left.\dfrac{d^n\left(\overline{u}\,(x,\alpha)\right)}{d\,x^n}\right|_{x=x_i} \end{cases} ,n\ is\ even$$

$$U_i(n,\alpha) = \begin{cases} \underline{U}_i(n,\alpha) = \overline{\varphi}\,(x_i,n,\alpha) = \left.\dfrac{d^n\left(\overline{u}\,(x,\alpha)\right)}{d\,x^n}\right|_{x=x_i} \\ \overline{U}_i(n,\alpha) = \underline{\varphi}\,(x_i,n,\alpha) = \left.\dfrac{d^n\left(\underline{u}(x,\alpha)\right)}{d\,x^n}\right|_{x=x_i} \end{cases} ,n\ is\ odd$$

where $\underline{U}_i(n,\alpha)$ and $\overline{U}_i(n,\alpha)$ are the lower and the upper spectrum of $u(x)$ at $x = x_i$ in the domain T, respectively.

If $u(x,\alpha)$ is $(i) - differentiable$, then $u(x,\alpha)$ can be represented as follows:

$$\begin{cases} \underline{u}(x,\alpha) = \sum_{n=0}^{\infty} \frac{(x-x_i)^n}{n!} \underline{U}(n,\alpha) \\ \overline{u}(x,\alpha) = \sum_{n=0}^{\infty} \frac{(x-x_i)^n}{n!} \overline{U}(n,\alpha) \end{cases} \quad , n \in N$$

and if $u(x,\alpha)$ is $(ii)-differentiable$, then $u(x,\alpha)$ can be represented as follows:

$$\begin{cases} \underline{u}(x,\alpha) = \sum_{n=0,even}^{\infty} \frac{(x-x_i)^n}{n!} \underline{U}(n,\alpha) + \sum_{n=1,odd}^{\infty} \frac{(x-x_i)^n}{n!} \overline{U}(n,\alpha) \\ \overline{u}(x,\alpha) = \sum_{n=0,even}^{\infty} \frac{(x-x_i)^n}{n!} \overline{U}(n,\alpha) + \sum_{n=1,odd}^{\infty} \frac{(x-x_i)^n}{n!} \underline{U}(n,\alpha) \end{cases}$$

(2.37)

equation (2.37) is known as the inverse transformation of $U(n,\alpha)$.

If $u(x,\alpha)$ is $(i)-differentiable$ then $U(n,\alpha)$ is defined as:

$$\underline{U}(n,\alpha) = W(n) \left[\frac{d^n \left(p(x)\underline{u}(x,\alpha) \right)}{dx^n} \right]_{x=x_0},$$

$$\overline{U}(n,\alpha) = W(n) \left[\frac{d^n \left(p(x)\overline{u}(x,\alpha) \right)}{dx^n} \right]_{x=x_0}.$$

if $u(x,\alpha)$ is $(ii)-differentiable$ then $U(n,\alpha)$ can be described as:

$$U(n,\alpha) = \begin{cases} \underline{U}(n,\alpha) = W(n) \left[\frac{d^n \left(p(x)\underline{u}(x,\alpha) \right)}{dx^n} \right]_{x=x_0} \\ \overline{U}(n,\alpha) = W(n) \left[\frac{d^n \left(p(x)\overline{u}(x,\alpha) \right)}{dx^n} \right]_{x=x_0} \end{cases} \quad , n \; is \; even$$

$$
U(n,\alpha)=\begin{cases}\underline{U}(n,\alpha)=W(n)\left[\dfrac{d^n\left(p(x)\overline{u}(x,\alpha)\right)}{dx^n}\right]_{x=x_0}\\ \overline{U}(n,\alpha)=W(n)\left[\dfrac{d^n\left(p(x)\underline{u}(x,\alpha)\right)}{dx^n}\right]_{x=x_0}\end{cases} \quad ,n\ is\ odd \tag{2.38}
$$

now, if $u(x,\alpha) is\ (i)-differentiable$ then $u(x,\alpha)$ can be represented as follows:

$$
u(x,\alpha)=\begin{cases}\underline{u}(x,\alpha)=\dfrac{1}{p(x)}\displaystyle\sum_{n=0}^{\infty}\dfrac{(x-x_0)^n}{n!}\dfrac{\underline{U}(n,\alpha)}{W(n)}\\ \overline{u}(x,\alpha)=\dfrac{1}{p(x)}\displaystyle\sum_{n=0}^{\infty}\dfrac{(x-x_0)^n}{n!}\dfrac{\overline{U}(n,\alpha)}{W(n)}\end{cases}
$$

and if $u(x,\alpha) is\ (ii)-differentiable$ then $u(x,\alpha)$ defined as:

$$
\begin{cases}\dfrac{1}{p(x)}\left(\displaystyle\sum_{n=0,even}^{\infty}\dfrac{(x-x_0)^n}{n!}\dfrac{\underline{U}(n,\alpha)}{W(n)}+\sum_{n=1,odd}^{\infty}\dfrac{(x-x_0)^n}{n!}\dfrac{\overline{U}(n,\alpha)}{W(n)}\right)\\ \dfrac{1}{p(x)}\left(\displaystyle\sum_{n=0,even}^{\infty}\dfrac{(x-x_0)^n}{n!}\dfrac{\overline{U}(n,\alpha)}{W(n)}+\sum_{n=1,odd}^{\infty}\dfrac{(x-x_0)^n}{n!}\dfrac{\underline{U}(n,\alpha)}{W(n)}\right)\end{cases} \tag{2.39}
$$

where $W(n)>0$ and it is the weighting factor and $p(x)>0$ is regard as a kernel corresponding to $u(x,\alpha)$. If $W(n)=1\ and\ p(x)=1$, then (2.37) is a special case of (2.39). in this section the transformation $W(n)=\frac{H^n}{n!}\ and\ p(x)=1$ will be applied, where H is the time horizon of interest. So, the equations (2.38) become

If $u(x,\alpha)$ is $(i)-differentiable$ then $U(n,\alpha)$ is defined as:

$$
\underline{U}(n,\alpha)=\frac{H^n}{n!}\frac{d^n\underline{u}(x,\alpha)}{dx^n},
$$

and

$$\overline{U}(n,\alpha) = \frac{H^n}{n!} \frac{d^n \overline{u}(x,\alpha)}{dx^n}$$

if $u(x,\alpha)$ is $(ii) - differentiable$ then $U(n,\alpha)$ can be described as:

$$U(n,\alpha) = \begin{cases} \underline{U}(n,\alpha) = \dfrac{H^n}{n!} \dfrac{d^n \underline{u}(x,\alpha)}{dx^n} \\ \overline{U}(n,\alpha) = \dfrac{H^n}{n!} \dfrac{d^n \overline{u}(x,\alpha)}{dx^n} \end{cases} \quad ,n\ is\ even$$

$$(2.40)$$

$$U(n,\alpha) = \begin{cases} \underline{U}(n,\alpha) = \dfrac{d^n \overline{u}(x,\alpha)}{dx^n} \\ \overline{U}(n,\alpha) = \dfrac{d^n \underline{u}(x,\alpha)}{dx^n} \end{cases} \quad ,n\ is\ odd$$

Using the fuzzy differential transformation, a fuzzy differential equation in the domain of interest can be transformed to an algebraic equation in the domain T, and $u(x)$ can be obtained as the finite- term Taylor series plus reminder, as:

if $u(x,\alpha) is\ (i) - differentiable$ then $u(x,\alpha)$ becomes:

$$u(x,\alpha) = \begin{cases} \underline{u}(x,\alpha) = \dfrac{1}{p(x)} \displaystyle\sum_{n=0}^{\infty} \dfrac{(x-x_0)^n}{n!} \dfrac{\underline{U}(n,\alpha)}{W(n)} + R_{n+1}(x) \\ \overline{u}(x,\alpha) = \dfrac{1}{p(x)} \displaystyle\sum_{n=0}^{\infty} \dfrac{(x-x_0)^n}{n!} \dfrac{\overline{U}(n,\alpha)}{W(n)} + R_{n+1}(x) \end{cases}$$

hence

$$u(x,\alpha)=\begin{cases}\underline{u}(x,\alpha)=\dfrac{1}{p(x)}\displaystyle\sum_{n=0}^{\infty}\left(\frac{x-x_0}{H}\right)^n\ \underline{U}(n,\alpha)+R_{n+1}(x)\\[2ex] \overline{u}(x,\alpha)=\dfrac{1}{p(x)}\displaystyle\sum_{n=0}^{\infty}\left(\frac{x-x_0}{H}\right)^n\overline{U}(n,\alpha)+R_{n+1}(x)\end{cases}\tag{2.41}$$

and, if $u(x,\alpha) is\ (ii)-differentiable$ then $u(x,\alpha)$ becomes:

$$\underline{u}(x,\alpha)=\frac{1}{p(x)}\left(\begin{array}{c}\displaystyle\sum_{n=0,even}^{\infty}\frac{(x-x_0)^n}{n!}\frac{\underline{U}(n,\alpha)}{W(n)}\\ +\displaystyle\sum_{n=1,odd}^{\infty}\frac{(x-x_0)^n}{n!}\frac{\overline{U}(n,\alpha)}{W(n)}\end{array}\right)+R_{n+1}(x),$$

$$\overline{u}(x,\alpha)=\frac{1}{p(x)}\left(\begin{array}{c}\displaystyle\sum_{n=0,even}^{\infty}\frac{(x-x_0)^n}{n!}\frac{\overline{U}(n,\alpha)}{W(n)}\\ +\displaystyle\sum_{n=1,odd}^{\infty}\frac{(x-x_0)^n}{n!}\frac{\underline{U}(n,\alpha)}{W(n)}\end{array}\right)+R_{n+1}(x)$$

hence,

$$\underline{u}(x,\alpha)=\frac{1}{p(x)}\left(\begin{array}{c}\displaystyle\sum_{n=0,even}^{\infty}\left(\frac{x-x_0}{H}\right)^n\ \underline{U}(n,\alpha)\\ +\displaystyle\sum_{n=1,odd}^{\infty}\left(\frac{x-x_0}{H}\right)^n\overline{U}(n,\alpha)\end{array}\right)+R_{n+1}(x),$$

$$\tag{2.42}$$

$$\overline{u}(x,\alpha)=\frac{1}{p(x)}\left(\begin{array}{c}\displaystyle\sum_{n=0,even}^{\infty}\left(\frac{x-x_0}{H}\right)^n\overline{U}(n,\alpha)\\ +\displaystyle\sum_{n=1,odd}^{\infty}\left(\frac{x-x_0}{H}\right)^n\ \underline{U}(n,\alpha)\end{array}\right)+R_{n+1}(x)$$

Our aim is finding the solution of the Volterra integral equation (1.17) at the equally spaced grid points $\{x_0, x_1, \dots, x_k\}$ where $x_0 = a, x_k = b,$

$x_i = a + ih\ for\ aech\ i = 0,1,2, \dots, k, and\ h = \frac{b-a}{k}$. This means, the domain of interest is divided to k sub-domain, and the fuzzy approximation functions in each sub-domain are $u_i(x, \alpha)$ for $i = 0,1 \dots, k-1$, respectively.

Definition $(\mathbf{2.10})[\mathbf{6}]$: (Fuzzy differential transformation)

Let $u(x, \alpha)$ be differentiable fuzzy-valued function, we define the one-dimensional differential transform by:

$$U(n,\alpha) = \begin{cases} \underline{U}(n,\alpha) = \dfrac{1}{n!}\left[\dfrac{d^n \underline{u}(x,\alpha)}{dx^n}\right]_{x=0} \\ \overline{U}(n,\alpha) = \dfrac{1}{n!}\left[\dfrac{d^n \overline{u}(x,\alpha)}{dx^n}\right]_{x=0} \end{cases}$$

where $u(x)$ is the original function and $U(n)$ is the transformed function.

Definition $(\mathbf{2.11})[\mathbf{6}]$: (Fuzzy differential inverse transformation) The differential inverse transform of $U(n, \alpha)$ on the girds x_{i+1} is defined by:

$$\begin{aligned} \underline{u}(x_{i+1},\alpha) &\approx \underline{u}_i(x_{i+1},\alpha) \\ &= \underline{U_i}(0,\alpha) + \underline{U_i}(1,\alpha)(t_1 - t_0) + \underline{U_i}(2,\alpha)(t_2 - t_1)^2 + \cdots \\ &+ \underline{U_i}(N,\alpha)(t_{i+1} - t_i)^N = \sum_{k=0}^{N} \underline{U_i}(k,\alpha) h^k, \end{aligned}$$

and

$$\overline{u}(x_{i+1},\alpha) \approx \overline{u}_i(x_{i+1},\alpha)$$
$$= \overline{U}_i(0,\alpha) + \overline{U}_i(1,\alpha)(t_1 - t_0) + \overline{U}_i(2,\alpha)(t_2 - t_1)^2 + \cdots$$
$$+ \overline{U}_i(N,\alpha)(t_{i+1} - t_i)^N = \sum_{k=0}^{N} \overline{U}_i(k,\alpha)h^k.$$

Theorem (2.7)[4]:

Let $u(x,\alpha), v(x,\alpha) and\, w(x,\alpha)$ are fuzzy valued functions, and let $U(x,\alpha), V(x,\alpha), and\, W(x,\alpha)$ are their differential transformations respectively, then we get the following:

i. If $u(x,\alpha) = v(x,\alpha) \pm w(x,\alpha), then\, U(n,\alpha) = V(n,\alpha) \pm W(n,\alpha)$.

ii. If $u(x,\alpha) = a\, v(x,\alpha), then\, U(n,\alpha) = a\, V(n,\alpha)$,where a is a constant.

iii. if
$$u(x,\alpha) = \frac{d^k v(x,\alpha)}{dx^k}\ , then\, U(n,\alpha) = \frac{(n+k)!}{n!} V(n+k,\alpha)$$

iv. If $u(x,\alpha) = x^k, then\, U(n,\alpha) = \delta(n-k,\alpha), where$
$$\delta(n-k) = \begin{cases} 1 & if\ n = k \\ 0 & if\ n \neq k \end{cases}$$

v. If
$$u(x,\alpha) = v(x,\alpha).w(x,\alpha), then\, U(n,\alpha) = \sum_{l=0}^{n} V(l).W(n-l,\alpha).$$

vi. If $u(x,\alpha) = \exp(a\,x,\alpha)\, , then\, U(n,\alpha) = \left({a^n}/{n!}, \alpha\right)\, , where$

$a\ is$ a constant.

vii. If

$$u(x,\alpha) = (x+1)^k, then\ U(n,\alpha) = \left(\frac{k(k-1)\ldots.(k-n-1)}{n!}, \alpha\right)$$

viii. If

$$u(x,\alpha) = (\sin(a\,x+b), \alpha), then\ U(n,\alpha) = \left(\frac{a^n}{n!}\sin(\pi\frac{n}{2!} + b\right),\alpha)$$

$where\ a, b\ are\ constants.$

ix. If

$$u(x,\alpha) = (\cos(a\,x+b), \alpha), then\ U(n,\alpha) = (\frac{a^n}{n!}\cos(\pi\frac{n}{2!} + b),\alpha)$$

where a, b are constants.

Proof: Using definition(2.9).

Theorem (2.8)[**6**]:

Suppose that $U(n,\alpha)$, $G(n,\alpha) and\ V(n)$ are differential transformations of the fuzzy- valued functions $u(x,\alpha)\ and\ g(x,\alpha)$, and positive real valued function $v(x)$, respectively then the following are satisfied:

(1) If $g(x,\alpha) = \int_{x_0}^{x} u(t,\alpha)dt,\ 0 \le \alpha \le 1, then$

$$G(n,\alpha) = \begin{cases} \underline{G}(n,\alpha) = \dfrac{\underline{U}(n-1,\alpha)}{n} \\ \overline{G}(n,\alpha) = \dfrac{\overline{U}(n-1,\alpha)}{n} \end{cases}, where\ n \ge 1.$$

(2) If $g(x,\alpha) = \int_{x_0}^{x} v(t)u(t,\alpha)dt,\ 0 \le \alpha \le 1, then$

$$G(n,\alpha) = \begin{cases} \underline{G}(n,\alpha) = \dfrac{1}{n}\displaystyle\sum_{l=0}^{n-1} V(l)\underline{U}(n-l-1,\alpha), \\ \overline{G}(n,\alpha) = \dfrac{1}{n}\displaystyle\sum_{l=0}^{n-1} V(l)\overline{U}(n-l-1,\alpha), \end{cases}, G(0) = \overline{0}, n \ge 1.$$

(3) If $g(x,\alpha) = v(x)\int_{x_0}^{x} u(t,\alpha)dt,\ 0 \le \alpha \le 1, then$

$$G(n,\alpha)=\begin{cases}\underline{G}(n,\alpha)=\sum_{l=1}^{n}\frac{1}{l}V(n-l)\underline{U}(l-1,\alpha),\\ \overline{G}(n,\alpha)=\sum_{l=1}^{n}\frac{1}{l}V(n-l)\overline{U}(l-1,\alpha),\end{cases},G(0)=\overline{0},n\geq 1.$$

(4) If $g(x,\alpha)=v(x)u(x,\alpha), then$

$$G(n,\alpha)=\sum_{l=0}^{n}V(l)\underline{U}(n-l,\alpha),$$

Proof: see [6].

now, taking the fuzzy differential transformation for both sides of equation (1.17), we get:

$$U(n,\alpha)=\begin{cases}\underline{U}(n,\alpha)=\ \underline{F}(n,\alpha)+\frac{\lambda}{n}\sum_{l=0}^{n-1}V(l)\underline{U}(n-l-1,\alpha)\\ \overline{U}(n,\alpha)=\ \overline{F}(n,\alpha)+\frac{\lambda}{n}\sum_{l=0}^{n-1}V(l)\overline{U}(n-l-1,\alpha)\end{cases}$$

where $\lambda>0$, $U(n,\alpha)$ *and* $F(n,\alpha)$are fuzzy differential transformations of fuzzy valued functions $u(x,\alpha),and\ f(x,\alpha)$ respectively, and $K(n)$ is a fuzzy differential transformation of a positive real- valued function $k(x)$. $and\ also\ U(0,\alpha)=\ F(0,\alpha)$.

Example (2.4):

Consider the fuzzy Volterra integral equation:

$$u(x)=(\alpha+1,3-\alpha)+\int_{0}^{x}(x-t)\,u(t)dt, x\in[0,T],T<\infty$$

Solution:

$$\underline{u}(x,\alpha) = (\alpha+1) + \int_{0}^{x} (x-t)\underline{u}\,(t,\alpha)dt,$$

$$\underline{u}(x,\alpha) = (3-\alpha) + \int_{0}^{x} (x-t)\overline{u}\,(t,\alpha)dt.$$

taking fuzzy differential transformation, we get:

$$\underline{U}(n,\alpha) = (\alpha+1)\delta(n-0) + \sum_{l=1}^{n} \frac{1}{l}\delta(n-l-1)\underline{U}(l-1,\alpha)$$

$$-\frac{1}{n}\sum_{l=0}^{n-1} \delta(l-1)\underline{U}(n-l-1,\alpha),$$

and

$$\overline{U}(n,\alpha) = (3-\alpha)\delta(n-0) + \sum_{l=0}^{n-1} \frac{1}{l}\delta(n-l-1)\overline{U}(l-1,\alpha)$$

$$-\frac{1}{n}\sum_{l=0}^{n-1} \delta(l-1)\overline{U}(n-l-1,\alpha).$$

First terms of differential transformation series are:

$\underline{U}(0,\alpha) = (\alpha+1)$

$$\underline{U}(1,\alpha) = (\alpha+1)\delta(1-0) + \sum_{l=1}^{1} \frac{1}{l}\delta(1-l-1)\underline{U}(l-1,\alpha)$$
$$-\frac{1}{1}\sum_{l=0}^{n-1} \delta(l-1)\underline{U}(1-l-1,\alpha) = \underline{0}$$

$$\underline{U}(2,\alpha) = (\alpha+1)\delta(2-0) + \sum_{l=1}^{2}\frac{1}{l}\delta(2-l-1)\underline{U}(l-1,\alpha) - \frac{1}{2}\sum_{l=0}^{1}\delta(l-1)\underline{U}(2-l-1,\alpha) = \frac{1}{2}\underline{U}(0,\alpha) = \frac{1}{2}(\alpha+1)$$

$$\underline{U}(3,\alpha) = (\alpha+1)\delta(3-0) + \sum_{l=1}^{3}\frac{1}{l}\delta(3-l-1)\underline{U}(l-1,\alpha) - \frac{1}{3}\sum_{l=0}^{2}\delta(l-1)\underline{U}(3-l-1,\alpha) = \underline{0}$$

$$\underline{U}(4,\alpha) = (\alpha+1)\delta(4-0) + \sum_{l=1}^{4}\frac{1}{l}\delta(4-l-1)\underline{U}(l-1,\alpha) - \frac{1}{4}\sum_{l=0}^{3}\delta(l-1)\underline{U}(4-l-1,\alpha) = \frac{1}{12}\underline{U}(2,\alpha) = \frac{1}{24}(\alpha+1)$$

$$\underline{U}(5,\alpha) = (\alpha+1)\delta(5-0) + \sum_{l=1}^{5}\frac{1}{l}\delta(5-l-1)\underline{U}(l-1,\alpha) - \frac{1}{5}\sum_{l=0}^{4}\delta(l-1)\underline{U}(5-l-1,\alpha) = \underline{0}$$

$$\underline{U}(6,\alpha) = (\alpha+1)\delta(5-0) + \sum_{l=1}^{6}\frac{1}{l}\delta(6-l-1)\underline{U}(l-1,\alpha) - \frac{1}{6}\sum_{l=0}^{5}\delta(l-1)\underline{U}(6-l-1,\alpha) = \frac{1}{30}\underline{U}(4,\alpha) = \frac{1}{720}(\alpha+1)$$

and

$$\overline{U}(0,\alpha) = (3-\alpha)$$

$$\overline{U}(1,\alpha) = (3-\alpha)\delta(1-0) + \sum_{l=1}^{1}\frac{1}{l}\delta(1-l-1)\overline{U}(l-1,\alpha) - \frac{1}{1}\sum_{l=0}^{n-1}\delta(l-1)\overline{U}(1-l-1,\alpha) = \overline{0}$$

$$\overline{U}(2,\alpha) = (3-\alpha)\delta(2-0) + \sum_{l=1}^{2}\frac{1}{l}\delta(2-l-1)\overline{U}(l-1,\alpha) - \frac{1}{2}\sum_{l=0}^{1}\delta(l-1)\overline{U}(2-l-1,\alpha) = \frac{1}{2}\overline{U}(0,\alpha) = \frac{1}{2}(3-\alpha)$$

$$\overline{U}(3,\alpha) = (3-\alpha)\delta(3-0) + \sum_{l=1}^{3}\frac{1}{l}\delta(3-l-1)\overline{U}(l-1,\alpha) - \frac{1}{3}\sum_{l=0}^{2}\delta(l-1)\overline{U}(3-l-1,\alpha) = \overline{0}$$

$$\overline{U}(4,\alpha) = (3-\alpha)\delta(4-0) + \sum_{l=1}^{4}\frac{1}{l}\delta(4-l-1)\overline{U}(l-1,\alpha) - \frac{1}{4}\sum_{l=0}^{3}\delta(l-1)\overline{U}(4-l-1,\alpha) = \frac{1}{12}\overline{U}(2,\alpha) = \frac{1}{24}(3-\alpha)$$

$$\overline{U}(5,\alpha) = (3-\alpha)\delta(5-0) + \sum_{l=1}^{5}\frac{1}{l}\delta(5-l-1)\overline{U}(l-1,\alpha) - \frac{1}{5}\sum_{l=0}^{4}\delta(l-1)\overline{U}(5-l-1,\alpha) = \overline{0}$$

$$\overline{U}(6,\alpha) = (3-\alpha)\delta(5-0) + \sum_{l=1}^{6}\frac{1}{l}\delta(6-l-1)\overline{U}(l-1,\alpha) - \frac{1}{6}\sum_{l=0}^{5}\delta(l-1)\overline{U}(6-l-1,\alpha) = \frac{1}{30}\overline{U}(4,\alpha) = \frac{1}{720}(3-\alpha)$$

therefore, the solution of fuzzy Volterra integral equation will be as following:

$$\underline{u}(x,\alpha) = \sum_{n=0}^{\infty}\underline{U}(n,\alpha).x^n$$

$$= (\alpha+1)\left\{1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\frac{1}{8!}x^8+\frac{1}{10!}x^{10}+\cdots\right\},$$

and

$$\overline{u}(x,\alpha) = \sum_{n=0}^{\infty}\overline{U}(n,\alpha).x^n$$

$$= (3-\alpha)\left\{1+\frac{1}{2!}x^2+\frac{1}{4!}x^4+\frac{1}{6!}x^6+\frac{1}{8!}x^8+\frac{1}{10!}x^{10}+\cdots\right\}.$$

this converges to the closed form solution:

$$u(x,\alpha) = \begin{cases}\underline{u}(x,\alpha) = (\alpha+1)\cosh(x)\\ \overline{u}(x,\alpha) = (3-\alpha)\cosh(x)\end{cases}$$

2.6 Fuzzy successive approximation method

Consider the following fuzzy Volterra integral equation:

$$\begin{cases} \underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^x \underline{k(x,t)u(t,\alpha)}\, dt \\ \overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_a^x \overline{k(x,t)u(t,\alpha)}\, dt \end{cases} \tag{2.43}$$

where $a \le t \le x, x \in [a,b], 0 \le \alpha \le 1, and$

$$\underline{k(x,t)u(t,\alpha)} = \begin{cases} k(x,t)\underline{u}(t,\alpha), k(x,t) \ge 0 \\ k(x,t)\overline{u}(t,\alpha), k(x,t) < 0 \end{cases}$$

and

$$\overline{k(x,t)u(t,\alpha)} = \begin{cases} k(x,t)\overline{u}(t,\alpha), k(x,t) \ge 0 \\ k(x,t)\underline{u}(t,\alpha), k(x,t) < 0 \end{cases}$$

for each $0 \le \alpha \le 1,\ and\ a \le x \le b.$

The successive approximation method introduces the recurrence relation:

$$\begin{cases} \underline{u}_{n+1}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^x k(x,t)\underline{u}_n(t,\alpha)dt\ \ , n \ge 0 \\ \overline{u}_{n+1}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_a^x k(x,t)\overline{u}_n(t,\alpha)dt\ \ , n \ge 0 \end{cases} \tag{2.44}$$

We can choose the $zeroth - component$ either $[\ \underline{0}, \overline{0}\ ], \left[\ \underline{f}(x,\alpha), \overline{f}(x,\alpha)\right]$

The exact solution $u(x,\alpha)\left[\ \underline{u}(x,\alpha), \overline{u}(x,\alpha)\right]$.

where

$$\lim_{n\to\infty} \underline{u}_{n+1} = \underline{u}(x,\alpha) \quad and \quad \lim_{n\to\infty} \overline{u}_{n+1} = \overline{u}(x,\alpha)$$

Consider the fuzzy Volterra integral equation:

$$u(x) = (\alpha + 1, 3 - \alpha) + \int_0^x (x-t)\, u(t)dt, x \in [0,T], T < \infty$$

Solution:

In this example $k(x,t) \geq 0, for\ each\ 0 \leq t \leq x$.

let $\underline{u}_0(x,\alpha) = (\alpha + 1)$

the first terms of successive approximation method series are:

$$\underline{u}_1(x,\alpha) = (\alpha + 1) + \int_0^x (x-t)\underline{u}_0(t,\alpha)dt$$

$$= (\alpha + 1) + \int_0^x (x-t)(\alpha + 1)dt$$

$$=(\alpha + 1)\left(1 + \frac{x^2}{2}\right)$$

$$\underline{u}_2(x,\alpha) = (\alpha + 1) + \int_0^x (x-t)\underline{u}_1(t,\alpha)dt$$

$$= (\alpha + 1) + \int_0^x (x-t)\left(1 + \frac{t^2}{2}\right)(\alpha + 1)dt$$

$$=(\alpha + 1)\left(1 + \frac{x^2}{2} + \frac{x^4}{24}\right)$$

$$\underline{u}_3(x,\alpha) = (\alpha+1) + \int_0^x (x-t)\underline{u}_2(t,\alpha)dt$$

$$= (\alpha+1) + \int_0^x (x-t)\left(1+\frac{t^2}{2}+\frac{t^4}{24}\right)(\alpha+1)dt$$

$$=(\alpha+1)\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}\right)$$

and

$$\overline{u}_0(x,\alpha) = (3-\alpha)$$

$$\overline{u}_1(x,\alpha) = (3-\alpha) + \int_0^x (x-t)\overline{u}_0(t,\alpha)dt$$

$$= (3-\alpha) + \int_0^x (x-t)(3-\alpha)dt$$

$$=(3-\alpha))\left(1+\frac{x^2}{2}\right)$$

$$\overline{u}_2(x,\alpha) = (3-\alpha) + \int_0^x (x-t)\overline{u}_1(t,\alpha)dt$$

$$= (3-\alpha) + \int_0^x (x-t)\left(1+\frac{t^2}{2}\right)(3-\alpha)dt$$

$$=(3-\alpha)\left(1+\frac{x^2}{2}+\frac{x^4}{24}\right)$$

$$\overline{u}_3(x,\alpha) = (3-\alpha) + \int_0^x (x-t)\overline{u}_2(t,\alpha)dt$$

$$= (3-\alpha) + \int_0^x (x-t)\left(1+\frac{t^2}{2}+\frac{t^4}{24}\right)(3-\alpha)dt$$

$$=(3-\alpha)\left(1+\frac{x^2}{2}+\frac{x^4}{24}+\frac{x^6}{720}\right)$$

hence, the exact solution is:

$$\underline{u}(x,\alpha) = (\alpha+1)\lim_{n\to\infty}\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)$$

$$= (\alpha+1)\cosh(x),$$

and

$$\overline{u}(x,\alpha) = \quad (3-\alpha)\lim_{n\to\infty}\left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)$$

$$= (3-\alpha)\cosh(x)$$

Chapter Three

Numerical Methods for Solving Linear Fuzzy Volterra Integral Equation of the Second Kind

Introduction

Several numerical methods for approximating linear Volterra integral equation is known. In this chapter we introduce three numerical methods to solve the Volterra fuzzy integral equation

$$u(x) = f(x) + \lambda \int_a^x k(x,t)u(t)dt$$

These methods are Taylor expansion method, trapezoidal method, and the variational iteration method.

3.1 Taylor Expansion Method

if $f(x)$ is continuous and infinity differentiable function on some open interval I, and if $x = z \in I$, then the taylor series of $f(x)$

about $x = z$ is given as:

$$f(x) = f(z) + \frac{1}{2!}f'(z)(x-z) + \cdots + \frac{1}{n!}f^{(n)}(z)(x-z)^n + \cdots$$

Here we use the Taylor expansion method to solve linear fuzzy Volterra integral equation of the second kind. This method depends on differentiating the fuzzy integral equation of the second kind $n\,times$, then substitute the Taylor series expansion for the unknown function. as a result, we get a linear system for which the solution of this system yields the unknown Taylor coefficient of the solution functions.

Definition$(\mathbf{3.1})[\mathbf{36}]$:

The second kind fuzzy Volterra integral equations system is in the form:

$$u_i(x) = f_i(x) + \sum_{j=1}^{m}\left(\lambda_{i,j}\int_a^x k_{i,j}(x,t)u_j(t)dt\right) \tag{3.1}$$

where $a \le t \le x \le b,\ and\ \lambda_{i,j} \ne 0\ \ for\ i,j = 1,2,\dots,m$ are real constants. Moreover, in system (3.1) the fuzzy function $f_i(x)$ and $k_{i,j}(x,t)$ are given, and assumed to be sufficiently differentiable with respect to all their arguments on the interval $a \le x \le t \le b$. also we assume that the kernel function $k_{i,j}(x,t) \in L^2([a,b]\times[a,b])$, and $u(x) = [u_1(x),\dots,u_m(x)]^T$ is the solution to be determined.

Now, the parametric form of $f_i(x)\ and\ \ u_i(x)$ are $\left(\underline{f_i}(x,\alpha),\overline{f}_i(x,\alpha)\right) and\ \left(\underline{u}_i(x,\alpha),\overline{u}_i(x,\alpha)\right)$, respectively. ($0 \le \alpha \le 1$,

$a \le x \le b$). To simplify we assume $\lambda_{i,j} > 0,\ (i,j = 1,2,\dots,m)$.

In order to design a numerical scheme for solving (3.1), we write the parametric form of the given fuzzy integral equations system as:

$$\begin{aligned}\underline{u}_i(x,\alpha) &= \underline{f_i}(x,\alpha) + \sum_{j=1}^{i}\left(\lambda_{i,j}\int_a^x \underline{u}_{i,j}(t,\alpha)dt\right)\\ \overline{u}_i(x,\alpha) &= \overline{f}_i(x,\alpha) + \sum_{j=1}^{i}\left(\lambda_{i,j}\int_a^x \overline{u}_{i,j}(t,\alpha)dt\right)\end{aligned} \tag{3.2}$$

where

$$\underline{u}_{i,j}(t,\alpha) = \begin{cases} k_{i,j}(x,t)\underline{u}_j(t,\alpha) & k_{i,j} \geq 0 \\ k_{i,j}(x,t)\overline{u}_j(t,\alpha) & k_{i,j} < 0 \end{cases}$$

$$\overline{u}_{i,j}(t,\alpha) = \begin{cases} k_{i,j}(x,t)\overline{u}_j(t,\alpha) & k_{i,j} \geq 0 \\ k_{i,j}(x,t)\underline{u}_j(t,\alpha) & k_{i,j} < 0 \end{cases} \tag{3.3}$$

now, we assume that

$$\begin{cases} \lambda_{i,j}k_{i,j}(x,t) \geq 0, & a \leq t \leq c_{i,j} \\ \lambda_{i,j}k_{i,j}(x,t) < 0, & c_{i,j} \leq t \leq x \end{cases} \tag{3.4}$$

then the system (3.2) becomes:

$$\begin{aligned} \underline{u}_i(x,\alpha) &= \underline{f_i}(x,\alpha) + \sum_{j=1}^{i} \lambda_{i,j} \left(\int_a^{c_{i,j}} k_{i,j}(x,t)\underline{u}_j(t,\alpha)dt + \int_{c_{i,j}}^{x} k_{i,j}(x,t)\overline{u}_j(t,\alpha)dt \right) \\ \overline{u}_i(x,\alpha) &= \overline{f}_i(x,\alpha) + \sum_{j=1}^{i} \lambda_{i,j} \left(\int_a^{c_{i,j}} k_{i,j}(x,t)\overline{u}_j(t,\alpha)dt + \int_{c_{i,j}}^{x} k_{i,j}(x,t)\underline{u}_j(t,\alpha)dt \right) \end{aligned} \tag{3.5}$$

We seek the solution of system (3.5) in the form of

$$\begin{aligned} \underline{u}_{j,s}(t,\alpha) &= \sum_{r=0}^{s} \left(\frac{1}{r!} \left. \frac{\partial^{(r)}\underline{u}_{js}(x,\alpha)}{r} \right|_{x=z} .(t-z)^r \right) \\ \overline{u}_{j,s}(t,\alpha) &= \sum_{r=0}^{s} \left(\frac{1}{r!} \left. \frac{\partial^{(r)}\overline{u}_{js}(x,\alpha)}{\partial x^r} \right|_{x=z} .(t-z)^r \right) \end{aligned} \quad a \leq x, z \leq b \tag{3.6}$$

for $j = 1, \dots, m$ which are the Taylor expansion of degree s at $x = z$

for the unknown functions $\underline{u}_{js}(x,\alpha), \overline{u}_{js}(x,\alpha)$, respectively.

To obtain the solution in the form of expression (3.6) we find the $n\,th$ derivative of each equation in the system (3.5) with respect to x by using the Leibnitz's rule, $(n = 0,1,\dots,s)$ and obtain by **[46]:**

$$\frac{\partial^{(n)}\underline{u}_{is}(x,\alpha)}{\partial x^n} = \frac{\partial^{(n)}\underline{f_i}(x,\alpha)}{\partial x^n} + \sum_{j=1}^{m} \lambda_{i,j} \left\{ \int_a^{c_{i,j}} \frac{\partial^{(n)}k_{i,j}(x,t)}{\partial x^n} \underline{u}_{js}(t,\alpha)dt \right.$$
$$\left. + \sum_{l=0}^{n-1} \left(\left. \frac{\partial^{(l)}k_{i,j}(x,t)}{\partial x^l} \right|_{t=x} . \overline{u}_{js}(x,\alpha) \right)^{(n-l-1)} + \int_{c_{i,j}}^{x} \frac{\partial^{(n)}k_{i,j}(x,t)}{\partial x^n} \overline{u}_{js}(t,\alpha)dt \right\}$$
$$\frac{\partial^{(n)}\overline{u}_{is}(x,\alpha)}{\partial x^n} = \frac{\partial^{(n)}\overline{f}_i(x,\alpha)}{\partial x^n} + \sum_{j=1}^{m} \lambda_{i,j} \left\{ \int_a^{c_{i,j}} \frac{\partial^{(n)}k_{i,j}(x,t)}{\partial x^n} \overline{u}_{js}(t,\alpha)dt \right.$$
$$\left. + \sum_{l=0}^{n-1} \left(\left. \frac{\partial^{(l)}k_{i,j}(x,t)}{\partial x^l} \right|_{t=x} . \overline{u}_{js}(x,\alpha) \right)^{(n-l-1)} + \int_{c_{i,j}}^{x} \frac{\partial^{(n)}k_{i,j}(x,t)}{\partial x^n} \underline{u}_{js}(t,\alpha)dt \right\}$$

(3.7)

for $n = 0,1,\dots,s$ $, and\ i = 1,2,\dots,m.$

From the Leibniz's rule, the system (3.7) becomes:

$$\frac{\partial^{(n)}\underline{u}_{is}(x,\alpha)}{\partial x^n} = \frac{\partial^{(n)}\underline{f_i}(x,\alpha)}{\partial x^n} + \sum_{j=1}^{m}\lambda_{i,j}\left\{\int_a^{c_{i,j}} \frac{\partial^{(n)}k_{i,j}(x,t)}{\partial x^n}\underline{u}_{js}(t,\alpha)dt\right.$$

$$+\sum_{r=0}^{n-1}\sum_{l=0}^{n-r-1}\binom{n-l-1}{r}\left(\left.\frac{\partial^{(l)}k_{i,j}(x,t)}{\partial x^l}\right|_{t=x}\right)^{(n-l-r-1)}\left(\overline{u}_{js}(x,\alpha)\right)^{(r)}$$

$$\left.+\int_{c_{i,j}}^{x}\frac{\partial^{(n)}k_{i,j}(x,t)}{\partial x^n}\overline{u}_{js}(t,\alpha)dt\right\}$$

$$\frac{\partial^{(n)}\overline{u}_{is}(x,\alpha)}{\partial x^n} = \frac{\partial^{(n)}\overline{f}_i(x,\alpha)}{\partial x^n} + \sum_{j=1}^{m}\lambda_{i,j}\left\{\int_a^{c_{i,j}} \frac{\partial^{(n)}k_{i,j}(x,t)}{\partial x^n}\overline{u}_{js}(t,\alpha)dt\right.$$

$$+\sum_{r=0}^{n-1}\sum_{l=0}^{n-r-1}\binom{n-l-1}{r}\left(\left.\frac{\partial^{(l)}k_{i,j}(x,t)}{\partial x^l}\right|_{t=x}\right)^{(n-l-r-1)}\left(\underline{u}_{js}(x,\alpha)\right)^{(r)}$$

$$+\int_{c_{i,j}}^{x}\frac{\partial^{(n)}k_{i,j}(x,t)}{\partial x^n}\underline{u}_{js}(t,\alpha)dt] \qquad (3.8)$$

our purpose is determining the coefficients $\underline{u}_{js}{}^{(n)}(z,\alpha)$ and $\overline{u}_{js}{}^{(n)}(z,\alpha), for\ n = 0,\dots,s, and\ j = 1,\dots,m$ in the system (3.7)

so that, we expand $\underline{u}_{js}(t,\alpha)$ and $\overline{u}_{js}(t,\alpha)$ in Taylor's series at arbitrary point $z: a \le z \le b$.

$$\begin{cases} \underline{u}_{j,s}(t,\alpha) = \sum_{r=0}^{s}\left(\frac{1}{r!}\left.\frac{\partial^{(r)}\underline{u}_{j,s}(x,\alpha)}{\partial x^r}\right|_{x=z}.(t-z)^r\right) \\ \overline{u}_{j,s}(t,\alpha) = \sum_{r=0}^{s}\left(\frac{1}{r!}\left.\frac{\partial^{(r)}\overline{u}_{j,s}(x,\alpha)}{\partial x^r}\right|_{x=z}.(t-z)^r\right) \end{cases} , a \le x,z \le b$$

$0 \le \alpha \le 1$, for $j = 1,\dots,m$ (3.9)

where

$$\underline{u}_{i,s}^{(n)}(z,\alpha) = \underline{f_i}^{(n)}(z,\alpha) + \sum_{j=1}^{m}\left\{\sum_{r=0}^{n-1} D_{n,r}^{(i,j)} \,.\, \overline{u}_{jS}^{(r)}(z,\alpha) + \sum_{r=0}^{s} E_{n,r}^{(i,j)} \,.\, \underline{u}_{jS}^{(r)}(z,\alpha) + \sum_{r=0}^{s} E'^{(i,j)}_{n,r} \,.\, \overline{u}_{jS}^{(r)}(z,\alpha)\right\}$$

$$\overline{u}_{i,s}^{(n)}(z,\alpha) = \overline{f}_i^{\,(n)}(z,\alpha) + \sum_{j=1}^{m}\left\{\sum_{r=0}^{n-1} D_{n,r}^{(i,j)} \,.\, \underline{u}_{jS}^{(r)}(z,\alpha) + \sum_{r=0}^{S} E_{n,r}^{(i,j)} \,.\, \overline{u}_{jS}^{(r)}(z,\alpha) + \sum_{r=0}^{s} E'^{(i,j)}_{n,r} \,.\, \underline{u}_{jS}^{(r)}(z,\alpha)\right\} \qquad (3.10)$$

then substitute the sth truncation in (3.8), we get

$$E_{n,r}^{(i,j)} = \frac{\lambda_{i,j}}{r!}\int_{a}^{c_{i,j}} \frac{\partial^{(n)}k_{i,j}}{\partial x^n}(x,t)\bigg|_{x=z} .\,(t-z)^r dt$$

$$E'^{(i,j)}_{n,r} = \frac{\lambda_{i,j}}{r!}\int_{c_{i,j}}^{z} \frac{\partial^{(n)}k_{i,j}}{\partial x^n}(x,t)\bigg|_{x=z} .\,(t-z)^r dt$$

$$D_{n,r}^{(i,j)} = \lambda_{i,j}\sum_{r=0}^{n-l}\binom{n-l-1}{r}\left(\frac{\partial^{(r)}k_{i,j}}{\partial x^r}(x,t)\bigg|_{x=z}\right)^{(n-l-r-1)}$$

$i,j = 1,2,\ldots,m$

$$for\ n = 0\,, \sum_{j=1}^{m}\sum_{r=0}^{n-1} D_{n,r}^{(i,j)}\,\overline{u}_{jS}^{(r)}(z,\alpha) = 0$$

$for\ \ n \le r, we\ have\quad D_{n,r}^{(i,j)} = 0, we\ take\ \ n,r = 0,1 \ldots s$

Consequently, the equation (3.8) can be written in the matrix form:

$$(D+E)U=\mathcal{F} \tag{3.11}$$

where

$$F=\begin{bmatrix} -\underline{f_1}(z,\alpha),\dots,-\underline{f_1}^{(S)}(x,\alpha)\Big|_{x=z},-\overline{f}_1(z,\alpha),\dots,-\overline{f}_1^{(S)}(x,\alpha)\Big|_{x=z},\dots, \\ -\underline{f_m}(z,\alpha),\dots,-\underline{f_m}^{(S)}(x,\alpha)\Big|_{x=z},-\overline{f}_m(z,\alpha),\dots,-\overline{f}_m^{(S)}(x,\alpha)\Big|_{x=z} \end{bmatrix}^t$$

$$u=\begin{bmatrix} \underline{u}_{1S}(z,\alpha),\dots,\underline{u}_{1S}^{(S)}(x,\alpha)\Big|_{x=z},\overline{u}_{1S}(z,\alpha),\dots,\overline{u}_{1S}^{(S)}(x,\alpha)\Big|_{x=z},\dots, \\ \underline{u}_{mS}(z,\alpha),\dots,\underline{u}_{mS}^{(S)}(x,\alpha)\Big|_{x=z},\overline{u}_{mS}(z,\alpha),\dots,\overline{u}_{mS}^{(S)}(x,\alpha)\Big|_{x=z} \end{bmatrix}^t$$

$$\mathrm{D}=\begin{bmatrix} D^{(1,1)} & \cdots & D^{(1,m)} \\ \vdots & \ddots & \vdots \\ D^{(m,1)} & \cdots & D^{(m,m)} \end{bmatrix}, \qquad \mathrm{E}=\begin{bmatrix} E^{(1,1)} & \cdots & E^{(1,m)} \\ \vdots & \ddots & \vdots \\ E^{(m,1)} & \cdots & E^{(m,m)} \end{bmatrix} \tag{3.12}$$

Parochial matrices $D^{(i,j)}$ $(\,for\; i,j=1,\dots m)$ are defined with the following elements:

$$D^{(i,j)}=\begin{bmatrix} D_{1,1}^{(i,j)} & D_{1,2}^{(i,j)} \\ D_{2,1}^{(i,j)} & D_{2,2}^{(i,j)} \end{bmatrix}, \quad E^{(i,j)}=\begin{bmatrix} E_{1,1}^{(i,j)} & E_{1,2}^{(i,j)} \\ E_{2,1}^{(i,j)} & E_{2,2}^{(i,j)} \end{bmatrix}, \quad i,j=\ 1,\dots,m \tag{3.13}$$

where

$$E_{1,1}^{(i,j)}=E_{2,2}^{(i,j)}=\begin{bmatrix} e_{0,0}^{(i,j)}-1 & e_{0,1}^{(i,j)} & \cdots & e_{0,s-1}^{(i,j)} & e_{0,s}^{(i,j)} \\ e_{1,0}^{(i,j)} & e_{1,1}^{(i,j)}-1 & \cdots & e_{1,s-1}^{(i,j)} & e_{1,s}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{s-1,1}^{(i,j)} & e_{s-1,1}^{(i,j)} & \cdots & e_{s-1,s-1}^{(i,j)}-1 & e_{s-1,s}^{(i,j)} \\ e_{s,0}^{(i,j)} & e_{s,1}^{(i,j)} & \cdots & e_{s,s-1}^{(i,j)} & e_{s,s}^{(i,j)}-1 \end{bmatrix},$$

$$E_{1,2}^{(i,j)} = E_{2,1}^{(i,j)} = \begin{bmatrix} e'^{(i,j)}_{0,0} - 1 & e'^{(i,j)}_{0,1} & \cdots & e'^{(i,j)}_{0,s-1} & e'^{(i,j)}_{0,s} \\ e'^{(i,j)}_{1,0} & e'^{(i,j)}_{1,1} - 1 & \cdots & e'^{(i,j)}_{1,s-1} & e'^{(i,j)}_{1,s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e'^{(i,j)}_{s-1,0} & e'^{(i,j)}_{s-1,1} & \cdots & e'^{(i,j)}_{s-1,s-1} - 1 & e'^{(i,j)}_{s-1,s} \\ e'^{(i,j)}_{s,0} & e'^{(i,j)}_{s,1} & \cdots & e'^{(i,j)}_{s,s-1} & e'^{(i,j)}_{s,s} - 1 \end{bmatrix}, \tag{3.14}$$

$$D_{1,2}^{(i,j)} = D_{2,1}^{(i,j)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ d'^{(i,j)}_{1,0} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d'^{(i,j)}_{s-1,0} & d'^{(i,j)}_{s-1,1} & \cdots & 0 & 0 \\ d'^{(i,j)}_{s,0} & d'^{(i,j)}_{s,1} & \cdots & d'^{(i,j)}_{s,s-1} & 0 \end{bmatrix}$$

$$D_{1,1}^{(i,j)} = D_{2,2}^{(i,j)} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(s+1)\times(s+1)}$$

3.2 Convergence Analysis

In this section, we will prove that the approximation solution from Taylor expansion method converges to the exact solution of fuzzy system (3.2).

Theorem $(\mathbf{3.1})$ $[\mathbf{36}]$: Let the kernel be bounded and belongs to L^2 and $\underline{u}_{j,s}(x,\alpha) and\, \overline{u}_{j,s}(x,\alpha)$ $(\,for\, j = 1, \dots, m)$ be Taylor polynomials of degree s, and their coefficients are computed by solving the linear system (3.12), then these polynomials converges to the exact solution of fuzzy system $(3.2), when\; s \to +\infty$.

Proof:

Consider the fuzzy system (3.2). Since the series (3.10) converges to $\underline{u}_j(x,\alpha) and\ \overline{u}_j(x,\alpha)(\ for\ j=1,\dots,m)$, respectively, $i.e$

$\underline{u}_j(x,\alpha)=\lim_{s\to\infty}\underline{u}_{js}(x,\alpha)\ ,\ \ and\ \overline{u}_j(x,\alpha)=\lim_{s\to\infty}\overline{u}_{js}(x,\alpha)$

then, we conclude that

$$\underline{u}_{is}(x,\alpha)=\underline{f_i}(x,\alpha)+\sum_{j=1}^{m}\lambda_{i,j}\left(\begin{array}{c}\displaystyle\int_{a}^{i}k_{i,j}(x,t)\underline{u}_j(t,\alpha)dt\\ \displaystyle+\int_{c_{i,j}}^{x}k_{i,j}(x,t)\overline{u}_j(t,\alpha)dt\end{array}\right),$$

$$\overline{u}_{is}(x,\alpha)=\overline{f}_i(x,\alpha)+\sum_{j=1}^{m}\lambda_{i,j}\left(\begin{array}{c}\displaystyle\int_{a}^{c_{i,j}}k_{i,j}(x,t)\overline{u}_j(t,\alpha)dt\\ \displaystyle+\int_{c_{i,j}}^{x}k_{i,j}(x,t)\underline{u}_j(t,\alpha)dt\end{array}\right)$$

$i,j=1,\dots,m$

$$(3.15)$$

now, we define the error function $e_s(x,\alpha)$as a difference between the two systems $(3.5)\ and\ (3.15)$, we obtain:

$$e_s(x,\alpha)=\sum_{i=1}^{m}e_{i,s}(x,\alpha), \qquad (3.16)$$

where, $\qquad e_{i,s}(x,\alpha)=\underline{e}_{i,s}(x,\alpha)+\overline{e}_{i,s}(x,\alpha),$

$$\underline{e}_{i,s}(x,\alpha) = \left(\underline{U}_i(x,\alpha) - \underline{U}_{i,s}(x,\alpha)\right) + \sum_{j=1}^{m} \lambda_{i,j} \left(\int_{a}^{c_{i,j}} k_{i,j}(x,t) \left(\underline{U}_j(t,\alpha)dt - \underline{U}_{j,s}(t,\alpha)\right) dt \right) + \sum_{j=1}^{m} \lambda_{i,j} \left(\int_{c_{i,j}}^{x} k_{i,j}(x,t) \left(\overline{U}_j(t,\alpha) - \overline{U}_{j,s}(t,\alpha)\right) dt \right) \tag{3.17}$$

we must show when $s \to +\infty,\ then\ e_s(x,\alpha) \to 0$

taking the $\|\quad\|$ of system (3.17) we proceed as:

$$\|e_s\| = \left\|\sum_{i=1}^{m} e_{is}\right\| \le \sum_{i=1}^{m} \|e_{is}\| = \sum_{i=1}^{m} \left\|\underline{e}_{is} + \overline{e}_{is}\right\| \le \sum_{i=1}^{m} \left(\left\|\underline{e}_{is}\right\| + \|\overline{e}_{is}\|\right)$$

$$= \sum_{i=1}^{m} \left\{ \left(\underline{u}_i(x,\alpha) - \underline{u}_{i,s}(x,\alpha)\right) + \sum_{j=1}^{m} \lambda_{i,j} \left(\int_{a}^{c_{i,j}} k_{i,j}(x,t) \left(\underline{u}_j(t,\alpha)dt - \underline{u}_{j,s}(t,\alpha)\right) dt \right) + \sum_{j=1}^{m} \lambda_{i,j} \left(\int_{c_{i,j}}^{x} k_{i,j}(x,t) \left(\overline{u}_j(t,\alpha) - \overline{u}_{j,s}(t,\alpha)\right) dt \right) \right\}$$

$$+ \left\| \left(\overline{u}_i(x,\alpha) - \overline{u}_{i,s}(x,\alpha)\right) + \sum_{j=1}^{m} \lambda_{i,j} \left(\int_{a}^{c_{i,j}} k_{i,j}(x,t) \left(\overline{u}_j(t,\alpha) - \overline{u}_{j,s}(t,\alpha)\right) dt \right) + \sum_{j=1}^{m} \lambda_{i,j} \left(\int_{c_{i,j}}^{x} k_{i,j}(x,t) \left(\underline{u}_j(t,\alpha) - \underline{u}_{j,s}(t,\alpha)\right) dt \right) \right\|$$

$$\leq \sum_{i=1}^{m} \left\| \left(\underline{u}_i(x,\alpha) - \underline{u}_{i,s}(x,\alpha) \right) + \left(\overline{u}_i(x,\alpha) - \overline{u}_{i,s}(x,\alpha) \right) \right\|$$

$$+ \sum_{i=1}^{m} \left\| \sum_{j=1}^{m} \lambda_{i,j} \int_{a}^{c_{i,j}} k_{i,j}(x,t)(\underline{u}_j(t,\alpha)dt - \underline{u}_{j,s}(t,\alpha) + \overline{u}_j(t,\alpha)dt - \overline{u}_{j,s}(t,\alpha))dt \right.$$

$$\left. + \sum_{j=1}^{m} \lambda_{i,j} \int_{c_{i,j}}^{x} k_{i,j}(x,t)(\overline{u}_j(t,\alpha) - \overline{u}_{j,s}(t,\alpha) + \underline{u}_j(t,\alpha) - \underline{u}_{j,s}(t,\alpha))dt \right\|$$

$$\leq \sum_{i=1}^{m} \left\| \left(\underline{u}_i(x,\alpha) - \underline{u}_{i,s}(x,\alpha) \right) + \left(\overline{u}_i(x,\alpha) - \overline{u}_{i,s}(x,\alpha) \right) \right\|$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{m} \left| \lambda_{i,j} \right| \left\| \int_{a}^{x} k_{i,j}(x,t) \, (\, \underline{u}_j(t,\alpha) - \underline{u}_{j,s}(t,\alpha) \right.$$

$$\left. + \overline{u}_j(t,\alpha)dt - \overline{u}_{j,s}(t,\alpha))dt \right\|$$

$$\leq \sum_{i=1}^{m} \left\| \left(\underline{u}_i(x,\alpha) - \underline{u}_{i,s}(x,\alpha) \right) + \left(\overline{u}_i(x,\alpha) - \overline{u}_{i,s}(x,\alpha) \right) \right\|$$

$$+ \sum_{i=1}^{m} \left\{ \sum_{j=1}^{m} \left| \lambda_{i,j} \right| \int_{a}^{x} \left\| k_{i,j}(x,t) \right\| (\left\| \underline{u}_j(t,\alpha) - \underline{u}_{j,s}(t,\alpha) \right\| + \left\| \overline{u}_j(t,\alpha) \right. \right.$$

$$\left. \left. - \overline{u}_{j,s}(t,\alpha) \right\|)dt \right\}$$

since $\left\| k_{i,j}(x,t) \right\|$ is continuous on $[a,b]$ $a \leq x \leq b$, therefore,

$\left\| k_{i,j}(x,t) \right\|$ is bounded, this implies that $\left\| \underline{u}_i(x,\alpha) - \underline{u}_{i,s}(x,\alpha) \right\| \to 0$,

and, $\left\| \overline{u}_i(x,\alpha) - \overline{u}_{i,s}(x,\alpha) \right\| \to 0 \; as \; s \to +\infty, hence \; \|e_s\| \to 0$.

3.3 Trapezoidal Method

We consider fuzzy Volterra integral equation of the second kind (1.18) and subdivide the interval of integration $[a, x]$ into s equal subintervals $[x_{i-1}, x_i]$ of width $L = \frac{b-a}{s}$, $s \geq 1$, where b is the end point we choose for x. Now let $x_i = a + iL = t_i$. $0 \leq i \leq s$

the Trapezoidal rule is:

$$\int_a^b f(x)dx = L\left\{\frac{f(a) + f(b)}{2} + \sum_{i=1}^{s-1} f(x_i)\right\}$$

using the trapezoidal rule with n subintervals, we approximate the fuzzy Volterra integral equation by **[20]:**

$$\int_a^x k(x,t)u(t)dt \approx L\left[\frac{k(x,t_0)u(t_0) + k(x,t_S)u(t_S)}{2} + \sum_{i=1}^{S-1} k(x,t_i)u(t_i)\right] \tag{3.18}$$

$$L = \frac{t_j - a}{j} = \frac{x_s - a}{s}, t_j \leq x, j \geq 1, b = x_S = t_S\,, \quad j = 0,1,\dots,s$$

let us define:

$$\begin{cases} \underline{G}_s(x,\alpha) = L\left[\frac{\underline{\underline{k(x,a)u(a)}} + \underline{\underline{k(x,b)u(b)}}}{2} + \sum_{i=1}^{n-1} \underline{k(x,t_i)u(t_i)}\right] \\ \overline{G}_s(x,\alpha) = L\left[\frac{\overline{k(x,a)u(a)} + \overline{k(x,b)u(b)}}{2} + \sum_{i=1}^{n-1} \overline{k(x,t_i)u(t_i)}\right] \end{cases} \tag{3.19}$$

hence, the fuzzy Volterra integral equation (1.18) is approximated by:

$$u(x,\alpha)=\begin{cases}\underline{f}(x,\alpha)+\underline{G}_s(\alpha)\\ \overline{f}(x,\alpha)+\overline{G}_s(\alpha)\end{cases} \tag{3.20}$$

we substitute $x=x_i$ in the equation (3.20), we get:

$$u(x_i,\alpha)=\begin{cases}\underline{f}(x_i,\alpha)+L\left[\frac{\underline{k(x_i,a)u(a)}+\underline{k(x_i,x_i)u(x_i)}}{2}+\sum_{m=1}^{i-1}\underline{k(x_i,x_m)u(x_m)}\right]\\ \overline{f}(x_i,\alpha)+L\left[\frac{\overline{k(x_i,a)u(a)}+\overline{k(x_i,x_i)u(x_i)}}{2}+\sum_{m=1}^{i-1}\overline{k(x_i,x_m)u(x_m)}\right]\end{cases}$$

$$(3.21)$$

now, we assume that

$$\begin{cases}k(x_i,x_m)\geq 0, & 0\leq m\leq c\\ k(x_i,x_m)\leq 0, & c+1\leq m\leq i\end{cases} \tag{3.22}$$

then, the equation (3.21) becomes:

$$\underline{u}(x_i,\alpha)=\underline{f}(x_i,\alpha)+\frac{L}{2}k(x_i,a)\underline{u}(a,\alpha)++L\sum_{m=1}^{c}k(x_i,x_m)\underline{u}(x_m)$$

$$+L\sum_{m=c+1}^{i}k(x_i,x_m)\overline{u}(x_m)+\frac{L}{2}k(x_i,x_i)\overline{u}(x_i,x_i),$$

and (3.23)

$$\overline{u}(x_i,\alpha)=\overline{f}(x_i,\alpha)+\frac{L}{2}k(x_i,a)\overline{u}(a,\alpha)++L\sum_{m=1}^{c}k(x_i,x_m)\overline{u}(x_m)$$

$$+L\sum_{m=c+1}^{i}k(x_i,x_m)\underline{u}(x_m)+\frac{L}{2}k(x_i,x_i)\underline{u}(x_i,x_i)$$

$i = 0,1,2,\dots,s$

from the equation (3.234) , the following system was obtained:

$$(W + V)\,U = F \tag{3.24}$$

where

$$u = \left[\underline{u}(x_0,\alpha),\underline{u}(x_1,\alpha),\dots,\underline{u}(x_s,\alpha),\overline{u}(x_0,\alpha),\overline{u}(x_1,\alpha),\dots,\overline{u}(x_s,\alpha)\right]^t$$

$$F = \left[\underline{f}(x_0,\alpha),\underline{f}(x_1,\alpha),\dots,\underline{f}(x_s,\alpha),\overline{f}(x_0,\alpha),\overline{f}(x_1,\alpha),\dots,\overline{f}(x_s,\alpha)\right]^t$$

$$W = \begin{bmatrix} W_{1,1} & W_{1,2} \\ W_{2,1} & W_{2,2} \end{bmatrix}$$

where $W_{1,1} = (w)_{i,j}$ $,0 \le i,j \le s$, $1 \le c \le s,$ with:

$$(w)_{i,j} = \begin{cases} -L\,k(x_i,x_j) & ,1 \le j \le i \le c \\ 1 & ,0 \le i = j \le s - \\ \dfrac{-L}{2}\,k(x_i,x_j) & ,1 \le i \le c, j = 0 \\ 0 & ,else\ where \end{cases}$$

$W_{1,1} = W_{2,2}$, $W_{1,2} = W_{2,1} = zero\ matrix$.

$$V = \begin{bmatrix} V_{1,1} & V_{1,2} \\ V_{2,1} & V_{2,2} \end{bmatrix}$$

where $V_{1,1} = (v)_{i,j}$ $,0 \le i,j \le s$, $1 \le c \le s$,with:

$$(v)_{i,j} = \begin{cases} -L\,k(x_i, x_j) & , c+1 \le j \le i \le s \\ \dfrac{-L}{2}\,k(x_i, x_j) & , 1 \le i = j \le s \\ \dfrac{-L}{2}\,k(x_i, x_j) & , c+1 \le i \le s, j = 0 \\ 0 & , else\ where \end{cases}$$

$V_{1,1} = V_{2,2}$, $V_{1,2} = V_{2,1} = zero\ matrix$.

for arbitrary fixed α we have

$$\underline{u}(x,\alpha) = \lim_{S\to\infty} \underline{G_n}(\alpha) = \int_a^x k(x,t)\underline{u}(t)dt,$$

and (3.25)

$$\underline{u}(x,\alpha) = \lim_{S\to\infty} \overline{G}_n(\alpha) = \int_a^x k(x,t)\overline{u}(t)dt.$$

Theorem $(\mathbf{3.2})$ $[\mathbf{46}]$:

If $f(x,\alpha)$ is continuous in the metric D, then $\underline{G_n}(\alpha)\ and\ \overline{G}_n(\alpha)$ converges uniformly in $\alpha\ to\ \underline{u}(x,\alpha)\ and\ \overline{u}(x,\alpha)$, respectively.

proof: see [46]

3.4 Variation Iteration Method

In this section, we will use the variation iteration method (VIM) to approximate the solution of Volterra integral equations of the second kind. The method constructs a convergent sequence of functions, which approximates the exact solution with some iteration.

Consider the following general nonlinear system by **[29]:**

$$L[u(x)] + N[u(t)] = h(x), \tag{3.26}$$

where L is a linear operator, N is a nonlinear operator and $h(x)$ is a given continuous function. from the Variational iteration method, we can construct the following correction functional:

$$u_{p+1}(x) = u_p(x) + \int_0^x \lambda(z)\left[L\left[u_p(z)\right] + N\left[\tilde{u}_p(z)\right] - h(z)\right] dz$$

(3.27)

where $p = 0,1,3, \ldots,$ and λ is a langrage multiplier which can identified optimally via variational theory, u_p is the $p\,th$ approximate solution, and $\tilde{u}_p$ is a restricted variation $\left(i.e, \delta\tilde{u}_p = 0\right)$.

now, let us differentiate both sides of the Volterra integral equation of the second kind with respect to x , we get:

$$u'(x) = f'(x) + \frac{d}{dx}\int_a^x k(x,t)u(t)dt \tag{3.28}$$

We use the variation iteration method in direction xThen we get the following iteration sequence:

$$u_{p+1}(x) =$$

$$u_p(x) + \int_0^x \lambda(z)\left[u_p'(z) - f'(z) - \frac{d}{dz}\int_a^z k(z,t)u_p(t)dt\right] dz$$

(3.29)

then we calculate variation with respect to u_p, notice that $\delta u_p(0)$=0, yields:

$$\delta u_{p+1} = \delta u_p + \lambda \delta u_p\big|_{z=x} - \int_0^x \lambda'(z)\delta u_p dz = 0 \qquad (3.30)$$

therefore, we get the following stationary conditions:

$$\lambda'(z)|_{z=x} = 0, \qquad 1 + \lambda(z)|_{z=x} = 0$$

hence, the lagrange multiplier, can be identified:

$$\lambda(z) = -1,$$

If we substitute the value of the the Lagrange multiplier into the equation (3.29), we obtain the following iteration formula:

$$u_{p+1}(x) = u_p(x) - \int_0^x \left[u_p'(z) - f'(z) - \frac{d}{dz}\int_a^z k(z,t)u_p(t)dt \right] dz \qquad (3.31)$$

we apply variational iteration method on fuzzy Volterra integral equation of the second kind:

$$\begin{cases} \underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^x \underline{k(x,t)u(t,\alpha)}dt \\ \overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_a^x \overline{k(x,t)u(t,\alpha)}dt \end{cases} \qquad (3.32)$$

we suppose that the kernel $k(x,t) > 0 \;\; for\; a \le t \le c\;, and$

$k(x,t) < 0\ for\ \ c \leq t \leq x, then\ the\ system$ (3.32) becomes by **[1]:**

$$\begin{cases} \underline{u}(x,\alpha) = \underline{f}(x,\alpha) + \lambda \int_a^c k(x,t)\underline{u}(t,\alpha)dt + \lambda \int_c^x k(x,t)\overline{u}(t,\alpha)dt \\ \overline{u}(x,\alpha) = \overline{f}(x,\alpha) + \lambda \int_a^c k(x,t)\overline{u}(t,\alpha)dt + \lambda \int_c^x k(x,t)\underline{u}(t,\alpha)dt \end{cases}$$

(3.33)

for each $0 \leq \alpha \leq 1\ and\ a \leq x \leq b,$

now, using variation iteration method and equation (3.31), we get the following iteration formulas:

$$\underline{u}_{p+1}(x,\alpha) = \underline{u}_p(x,\alpha) - \int_0^x \left[\underline{u}_p'(z,\alpha) - \underline{f}'(z,\alpha) - \int_a^C \frac{\partial k(z,t)}{\partial z} \underline{u}_p(t,\alpha)dt - k(z,z)\overline{u}_p(z,\alpha) - \int_c^z \frac{\partial k(z,t)}{\partial z}(t,\alpha)\overline{u}_p(t,\alpha)dt \right] dz$$

(3.34)

and

$$\overline{u}_{p+1}(x,\alpha) = \overline{u}_p(x,\alpha) - \int_0^x \left[\overline{u}_p'(z,\alpha) - \overline{f}'(z,\alpha) - \int_a^C \frac{\partial k(z,t)}{\partial z} \overline{u}_p(t,\alpha)dt - k(z,z)u(z,\alpha) - \int_c^z \frac{\partial k(z,t)}{\partial z}(t,\alpha)\underline{u}_k(t,\alpha)dt \right] dz$$

where $p = 0,1,2, \ldots$

Using the formulas in (3.34), we can find a solution of equation (3.33) and hence obtain a fuzzy solution of the linear fuzzy Volterra integral equation of the second kind.

Chapter Four

Numerical Examples and Results

In this chapter, we will use algorithms and MAPLE software to solve numerical examples, then draw a comparison between approximate solutions and exact ones, to demonstrate the efficiency of these numerical schemes in chapter three. These include: Taylor expansion method, Trapezoidal method, and variation iteration method.

Numerical example $(\mathbf{4.1})$: $(\boldsymbol{Taylor\ expansion\ method})$

Consider the linear fuzzy Volterra integral equations:

$$\begin{cases} \underline{u}(x,\alpha) = (\alpha+1)x^2 + \int_0^x (x-t)\underline{u}(t,\alpha)dt \\ \overline{u}(x,\alpha) = (3-\alpha)x^2 + \int_0^x (x-t)\overline{u}(t,\alpha)dt \end{cases} \tag{4.1}$$

$0 \le x \le 1, 0 \le \alpha \le 1$

the analytical solution of the above problem is given by:

$$\begin{cases} \underline{u}(x,\alpha) = (\alpha+1)\,(e^x + e^{-x} - 2) \\ \overline{u}(x,\alpha) = (3-\alpha)(e^x + e^{-x} - 2) \end{cases} \quad 0 \le \alpha \le 1 \tag{4.2}$$

we expand the unknown functions in the Taylor series at $z = \frac{1}{4}$

to solve the previous example by using Taylor expansion method using MAPLE software, we will apply the following algorithm:

Algorithm (4.1)

1. Input $a, b, z, m,\ \lambda\ , k_{i,j}(x,t), \underline{f_i}(x,\alpha), \overline{f_i}(x,\alpha)$
2. Input the Taylor expansion degree (s)
3. Calculate

$$\frac{\partial^{(n)} k_{i,j}(x,\alpha)}{\partial x^n}, \frac{\partial^{(n)} \underline{f_i}(x,\alpha)}{\partial x^n}, \frac{\partial^{(n)} \overline{f}_i(x,\alpha)}{\partial x^n} \quad , n = 0,1, \dots, s$$

4. Calculate

$$e_{n+1,r+1}^{(i,j)} = \frac{\lambda_{i,j}}{r!} \int_a^{c_{i,j}} \frac{\partial^r k_{i,j}}{\partial x^r}(x,t).(t-z)^r\, dt \qquad i,j = 1, \dots, m$$

5. Calculate

$$e'^{(i,j)}_{n+1,r+1} = \frac{\lambda_{i,j}}{r!} \int_a^{c_{i,j}} \frac{\partial^r k_{i,j}}{\partial x^r}(x,t).(t-z)^r\, dt \qquad i,j = 1, \dots, m$$

6. Put

$$E_{1,1}^{(i,j)} = E_{2,2}^{(i,j)}$$

$$= \begin{bmatrix} e_{1,1}^{(i,j)} - 1 & e_{1,2}^{(i,j)} & \cdots & e_{1,s}^{(i,j)} & e_{1,s+1}^{(i,j)} \\ e_{2,1}^{(i,j)} & e_{2,2}^{(i,j)} - 1 & \cdots & e_{2,s}^{(i,j)} & e_{2,s+1}^{(i,j)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{s,1}^{(i,j)} & e_{s,2}^{(i,j)} & \cdots & e_{s,s}^{(i,j)} - 1 & e_{s,s+1}^{(i,j)} \\ e_{s+1,1}^{(i,j)} & e_{s+1,2}^{(i,j)} & \cdots & e_{s+1,s}^{(i,j)} & e_{s+1,s+1}^{(i,j)} - 1 \end{bmatrix}_{(s+1)\times(s+1)}$$

7. $E'^{(i,j)}_{1,2} = E'^{(i,j)}_{2,1}$

$$= \begin{bmatrix} e'^{(i,j)}_{1,1} - 1 & e'^{(i,j)}_{1,2} & \cdots & e'^{(i,j)}_{1,s} & e'^{(i,j)}_{1,s+1} \\ e'^{(i,j)}_{2,1} & e'^{(i,j)}_{2,2} - 1 & \cdots & e'^{(i,j)}_{2,s} & e'^{(i,j)}_{2,s+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e'^{(i,j)}_{s,1} & e'^{(i,j)}_{s,2} & \cdots & e'^{(i,j)}_{s,s} - 1 & e'^{(i,j)}_{s,s+1} \\ e'^{(i,j)}_{s+1,1} & e'^{(i,j)}_{s+1,2} & \cdots & e'^{(i,j)}_{s+1,s} & e'^{(i,j)}_{s+1,s+1} - 1 \end{bmatrix}$$

where $i,j=1,2,\dots,m$

8. Calculate

$$d'_{n+1,l+1}=\sum_{r=0}^{n-l-1}\binom{n-r-1}{n-l-r-1}\left(\left.\frac{\partial^{(r)}k_{i,j}(x,t)}{\partial x^{r}}\right|_{t=x}\right)^{(n-l-r-1)}\Bigg|_{x=z}$$

9. Put

$$D_{1,1}^{(i,j)}=D_{2,2}^{(i,j)}=\begin{bmatrix}0&0&\cdots&0&0\\0&0&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&0&0\\0&0&\cdots&0&0\end{bmatrix}_{(s+1)\times(s+1)}$$

10. Put

$$D'^{(i,j)}_{1,2}=D'^{(i,j)}_{2,1}=\begin{bmatrix}0&0&\cdots&0&0\\d'^{(i,j)}_{2,1}&0&\cdots&0&0\\\vdots&\vdots&\ddots&\vdots&\vdots\\d'^{(i,j)}_{k,1}&d'^{(i,j)}_{k,2}&\cdots&0&0\\d'^{(i,j)}_{k+1,1}&d'^{(i,j)}_{k+1,2}&\cdots&d'^{(i,j)}_{k+1,k}&0\end{bmatrix}_{(s+1)\times(s+1)}$$

11. denote $D^{(i,j)}=\begin{bmatrix}D_{1,1}^{(i,j)}&D_{1,2}^{(i,j)}\\D_{2,1}^{(i,j)}&D_{2,2}^{(i,j)}\end{bmatrix}$,

12. put

$$\mathcal{F}=\begin{bmatrix}-\underline{f}_1(z,\alpha),\dots,-\underline{f}_1^{(k)}(z,\alpha),-\overline{f}_1(z,\alpha),\dots,-\overline{f}_1^{(s)}(z,\alpha),\dots,\\-\underline{f}_m(z,\alpha),\dots,-\underline{f}_m^{(s)}(z,\alpha),-\overline{f}_m(z,\alpha),\dots,-\overline{f}_m^{(s)}(z,\alpha)\end{bmatrix}^{t}$$

13. put

$$u=\begin{bmatrix}\underline{u}_{1s}(z,\alpha),\dots,\underline{u}_{1s}^{(s)}(z,\alpha),\overline{u}_{1s}(z,\alpha),\dots,\overline{u}_{1s}^{(s)}(z,\alpha),\dots,\\ \underline{u}_{ms}(z,\alpha),\dots,\underline{u}_{ms}^{(s)}(z,\alpha),\overline{u}_{ms}(z,\alpha),\dots,\overline{u}_{ms}^{(s)}(z,\alpha)\end{bmatrix}^t$$

14. solve the following linear system $(E+D)u=\mathcal{F}$

15. estimate $\underline{u}(z,\alpha),\overline{u}(z,\alpha)$ by computing Taylor expansion for u

$$\begin{cases}\underline{u}_{j,s}=\sum_{r=0}^{s}\left(\frac{1}{r!}\left.\frac{\partial^{(r)}\underline{u}_j(x,\alpha)}{\partial x^r}\right|_{x=z}(x-z)^r\right)\\ u_{j,s}=\sum_{r=0}^{s}\left(\frac{1}{r!}\left.\frac{\partial^{(r)}\overline{u}_j(x,\alpha)}{\partial x^r}\right|_{x=z}(x-z)^r\right)\end{cases},\quad a\le x,z\le b,$$

$0\le\alpha\le1,\quad j=1,2,\dots,m$

We get the following results

$$E_{1,1}^{(1,1)}=E_{2,2}^{(1,1)}=\begin{bmatrix}-0.96875 & -0.00520833 & 0.00048828 & -0.00003255\\ 0.25000 & -1.03125000 & 0.00260416 & -0.00016276\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$$

$$E_{1,2}^{(1,1)}=E_{2,1}^{(1,1)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

hence,

$$E=\begin{bmatrix}E_{1,1}^{(1,1)} & E_{1,2}^{(1,1)}\\ E_{2,1}^{(1,1)} & E_{2,2}^{(1,1)}\end{bmatrix}$$

$$D_{1,1}^{(1,1)}=D_{2,2}^{(1,1)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\1&0&0&0\\0&1&0&0\end{bmatrix}$$

$$D_{1,2}^{(1,1)}=D_{2,1}^{(1,1)}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$D=\begin{bmatrix} D_{1,1}^{(1,1)} & D_{1,2}^{(1,1)} \\ D_{2,1}^{(1,1)} & D_{2,2}^{(1,1)} \end{bmatrix}=\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 1&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0 \end{bmatrix}$$

$$\mathcal{F}(z,\alpha)=\begin{bmatrix} -\underline{f}(z,\alpha),-\underline{f}'(z,\alpha),\underline{-f}''(z,\alpha),\underline{-f}'''(z,\alpha), \\ -\overline{f}(z,\alpha),\overline{-f}'(z,\alpha),-\overline{f}''(z,\alpha),-\overline{f}'''(z,\alpha) \end{bmatrix}^t$$

$$\mathcal{F}(z,\alpha)=$$

$$\left[\frac{-1}{16}(\alpha+1),\frac{-1}{2}(\alpha+1),-2(\alpha+1),0,\frac{-1}{16}(3-\alpha),\frac{-1}{2}(3-\alpha),-2(3-\alpha),0\right]^t$$

Solving the linear system

$$(D+E)u=\mathcal{F}$$

we get the following:

$$u(z,\alpha)=\begin{bmatrix} 0.0628227120219095(\alpha+1) \\ 0.505207645425615(\alpha+1) \\ 2.06282271202191(\alpha+1) \\ 0.505207645425615(\alpha+1) \\ 0.0628227120219095(3-\alpha) \\ 0.505207645425615(3-\alpha) \\ 2.06282271202191(3-\alpha) \\ 0.505207645425615(3-\alpha) \end{bmatrix}$$

$$\begin{cases} \underline{u}(x,\alpha)=\displaystyle\sum_{r=0}^{s}\left(\frac{1}{r!}\left.\frac{\partial^{(r)}\underline{u}_j(x,\alpha)}{\partial x^r}\right|_{x=z}\left(x-\frac{1}{4}\right)^r\right) \\ \overline{u}(x,\alpha)=\displaystyle\sum_{r=0}^{s}\left(\frac{1}{r!}\left.\frac{\partial^{(r)}\overline{u}_j(x,\alpha)}{\partial x^r}\right|_{x=z}\left(x-\frac{1}{4}\right)^r\right) \end{cases},\quad 0\le x\le 1, 0\le\alpha\le 1$$

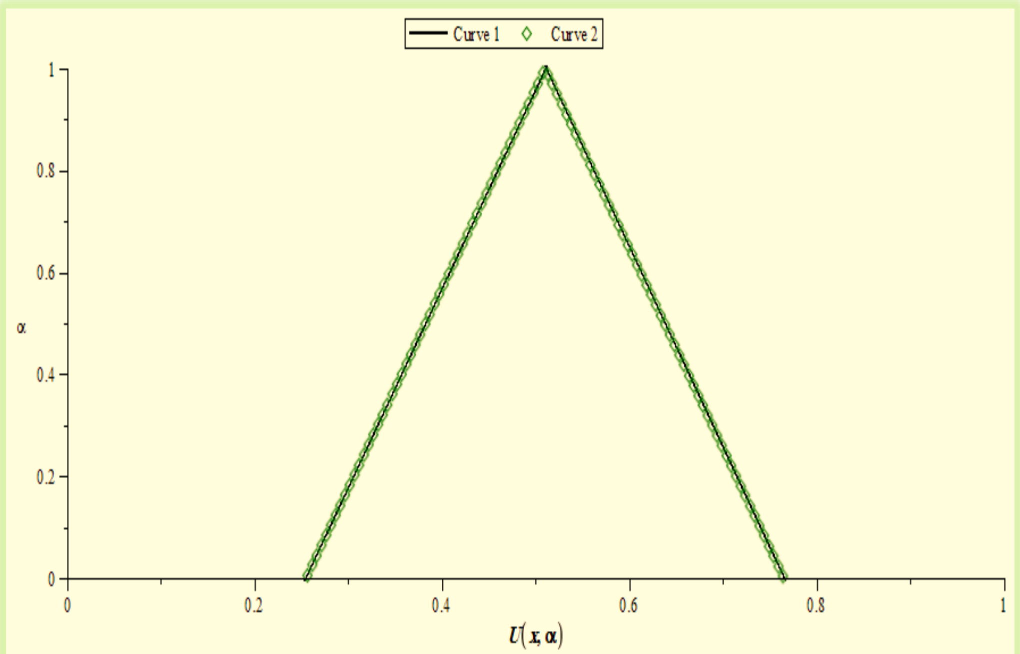
hence, the approximated solution is:

$$\underline{u}(x,\alpha) = 2.000407015\,x + 1.738708618\,\alpha + .3524026447x^3 + 0.1975243281\,x^4 + 0.1905848255\,x^5 - 0.73872694 + 0.04282267662\,\alpha\,x^4 - 0.01020620726\,\alpha\,x^5 - 1.000007782\,\alpha\,x + 0.9961625156\,x^2 - 0.4999246903\,\alpha\,x^2 + 0.1662755673\,\alpha\,x^3,$$

and

$$\overline{u}(x,\alpha) = 0.18846813606572846 - 0.06282271202190949\alpha + (3.09423406803286 - 1.0314113560109548\,\alpha)\left(x-\frac{1}{4}\right)^2 + (1.5156229362768463 - 0.5052076454256154\,\alpha)\left(x-\frac{1}{4}\right) + (0.2526038227128076 - 0.08420127423760254\alpha)\left(x-\frac{1}{4}\right)^3$$

Figure (4.1) (a) compares between the exact and numerical solutions for $u(x,\alpha)$ in Example (4.1) at $x = \frac{1}{2}$



Figure(4.1) (a) the exact and approximate solution at x=0.5

Figure (4.1) (a) shows the absolute error between the exact and numerical solutions for $u(x,\alpha)$ in Example (4.1) at $x = \frac{1}{2}$

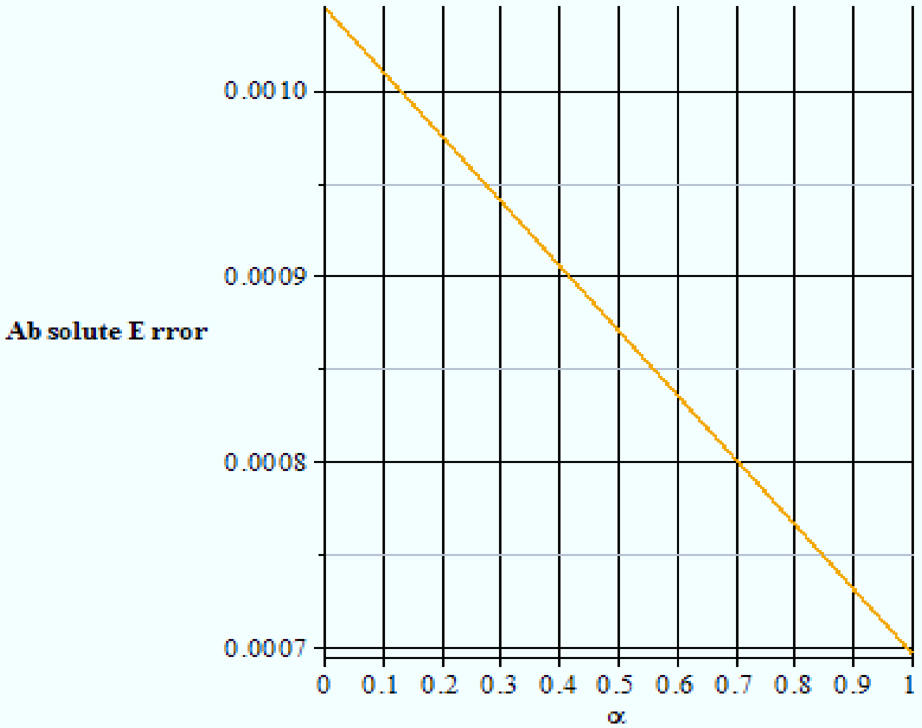


Figure (4.1)(b) absolute error between exact and numerical solutions at x=0.5

Table $(\mathbf{4.1})(\mathbf{b})$ shows the comparison between the exact solution and the approximate solution, and the resulted error.

Table 4.1: The exact and numerical solutions for $u(x,\alpha)$, and the resulted error by algorithm (4.1) $at\ x = \frac{1}{2}$

α	Numerical solution $\underline{u}(x,\alpha)$	Exact solution $u(x,\alpha)$	Numerical solution $\overline{u}(x,\alpha)$	Exact solution $\overline{u}(x,\alpha)$	Error =D (u_{exact}, u_{app})
0	0.2549034780	0.255251931	0.764710434	0.765755792	1.045×10^{-3}
0.1	0.2803938258	0.280777124	0.739220086	0.740230599	1.0105×10^{-3}
0.2	0.3058841736	0.306302317	0.713729738	0.714705406	9.7566×10^{-4}
0.3	0.3313745214	0.331827510	0.688239390	0.689180213	9.4082×10^{-4}
0.4	0.3568648692	0.357352703	0.662749042	0.663655020	9.0598×10^{-4}
0.5	0.3823552170	0.382877896	0.637258695	0.638129827	8.7113×10^{-4}
0.6	0.4078455648	0.408403090	0.611768347	0.612604633	8.3628×10^{-4}
0.7	0.4333359126	0.433928282	0.5862779994	0.587079440	8.0144×10^{-4}
0.8	0.4588262604	0.459453475	0.560787651	0.561554247	7.6659×10^{-4}
0.9	0.4843166082	0.484978668	0.535297303	0.536029054	7.3175×10^{-4}
1.0	0.5098069560	0.510503861	0.509806956	0.510503861	6.9690×10^{-4}

These results show the accuracy of the Taylor expansion method to solve the equation (4.1) with maximum error

$$= 1.045 \times 10^{-3}.$$

Numerical example $(\mathbf{4.2})$: $(\boldsymbol{Taylor\ expansion\ method})$

Consider the linear fuzzy Volterra integral equations:

$$\begin{cases} \underline{u}(x,\alpha) = (\alpha)\cot(x) + \int_0^x e^{x-t}\underline{u}(t,\alpha)dt \\ \overline{u}(x,\alpha) = (2-\alpha)\cot(x) + \int_0^x e^{x-t}\overline{u}(t,\alpha)dt \end{cases} \quad 0 \le \alpha \le 1 \qquad (4.3)$$

$x \in \left[0\ ,\frac{\pi}{4}\right]$

The analytical solution of the above problem is given by,

$$\begin{cases} \underline{u}(x,\alpha) = (\alpha)\left(\frac{3}{5}\cos(x) + \frac{1}{5}\sin(x) + \frac{2}{5}e^{2x}\right) \\ \overline{u}(x,\alpha) = (2-\alpha)\left(\frac{3}{5}\cos(x) + \frac{1}{5}\sin(x) + \frac{2}{5}e^{2x}\right) \end{cases} \qquad (4.4)$$

$0 \le \alpha \le 1$

Expand the unknown functions in Taylor series at $z = \frac{\pi}{12}$

To solve the previous example by using the Taylor expansion method using MAPLE software, we will apply algorithm(4.1) :

$$E_{1,1}^{(1,1)} = E_{2,2}^{(1,1)} =$$

$$\begin{bmatrix} -.70073 & -.04088 & .00364 & -.00024 & 1.275\times10^{-5} & -5.598\times10^{-7} \\ .29927 & -1.0409 & .00364 & -.00024 & 1.275\times10^{-5} & -5.598\times10^{-7} \\ .29927 & -.04088 & -.99636 & -.00024 & 1.275\times10^{-5} & -5.598\times10^{-7} \\ .29927 & -.04088 & .00364 & -1.0002 & 1.275\times10^{-5} & -5.598\times10^{-7} \\ .29927 & -.04088 & .00364 & -.00024 & -0.99999 & -5.598\times10^{-7} \\ .29927 & -.04088 & .00364 & -.00024 & 1.275\times10^{-5} & -1.000 \end{bmatrix}$$

$$E_{1,2}^{(1,1)}=E_{2,1}^{(1,1)}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

hence,

$$E=\begin{bmatrix}E_{1,1}^{(1,1)}&E_{1,2}^{(1,1)}\\E_{2,1}^{(1,1)}&E_{2,2}^{(1,1)}\end{bmatrix}$$

$$D_{1,1}^{(1,1)}=D_{2,2}^{(1,1)}=\begin{bmatrix}0&0&0&0&0&0\\1&0&0&0&0&0\\1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&0&0\\1&1&1&1&1&0\end{bmatrix}$$

$$D_{1,2}^{(1,1)}=D_{2,1}^{(1,1)}=\begin{bmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

$$D=\begin{bmatrix}D_{1,1}^{(1,1)}&D_{1,2}^{(1,1)}\\D_{2,1}^{(1,1)}&D_{2,2}^{(1,1)}\end{bmatrix}$$

$$\mathcal{F}(z,\alpha)=\begin{bmatrix}-\underline{f}(z,\alpha),-\underline{f}'(z,\alpha),-\underline{f}''(z,\alpha),-\underline{f}'''(z,\alpha),-\underline{f}^{(4)}(z,\alpha),-\underline{f}^{(5)}(z,\alpha)\\-\overline{f}(z,\alpha),-\overline{f}'(z,\alpha),-\overline{f}''(z,\alpha),-\overline{f}'''(z,\alpha),-\overline{f}^{(4)}(z,\alpha),-\overline{f}^{(5)}(z,\alpha)\end{bmatrix}^t$$

$$\boldsymbol{\mathcal{F}} = \begin{bmatrix} -\alpha\cos\left(\frac{\pi}{12}\right) \\ \alpha\, sin\left(\frac{\pi}{12}\right) \\ \alpha\cos\left(\frac{\pi}{12}\right) \\ -\alpha\, sin\left(\frac{\pi}{12}\right) \\ -\alpha\cos\left(\frac{\pi}{12}\right) \\ \alpha\, sin\left(\frac{\pi}{12}\right) \\ -(2-\alpha)\cos\left(\frac{\pi}{12}\right) \\ (2-\alpha)\, sin\left(\frac{\pi}{12}\right) \\ (2-\alpha)\cos\left(\frac{\pi}{12}\right) \\ -(2-\alpha)\, sin\left(\frac{\pi}{12}\right) \\ -(2-\alpha)\cos\left(\frac{\pi}{12}\right) \\ (2-\alpha)\, sin\left(\frac{\pi}{12}\right) \end{bmatrix}$$

Solving the linear system

$$(D+E)u = \mathcal{F}$$

we get the following:

$$u(z,\alpha)=\begin{bmatrix} 1.30655494406977\alpha \\ 1.38836501660070\alpha \\ 2.06962325214024\alpha \\ 5.36399137514409\alpha \\ 11.4350895320246\alpha \\ 21.4350895320246\alpha \\ 1.30655494406977(2-\alpha) \\ 1.38836501660070(2-\alpha) \\ 2.06962325214024(2-\alpha) \\ 5.36399137514409(2-\alpha) \\ 11.4350895320246(2-\alpha) \\ 21.4350895320246(2-\alpha) \end{bmatrix}$$

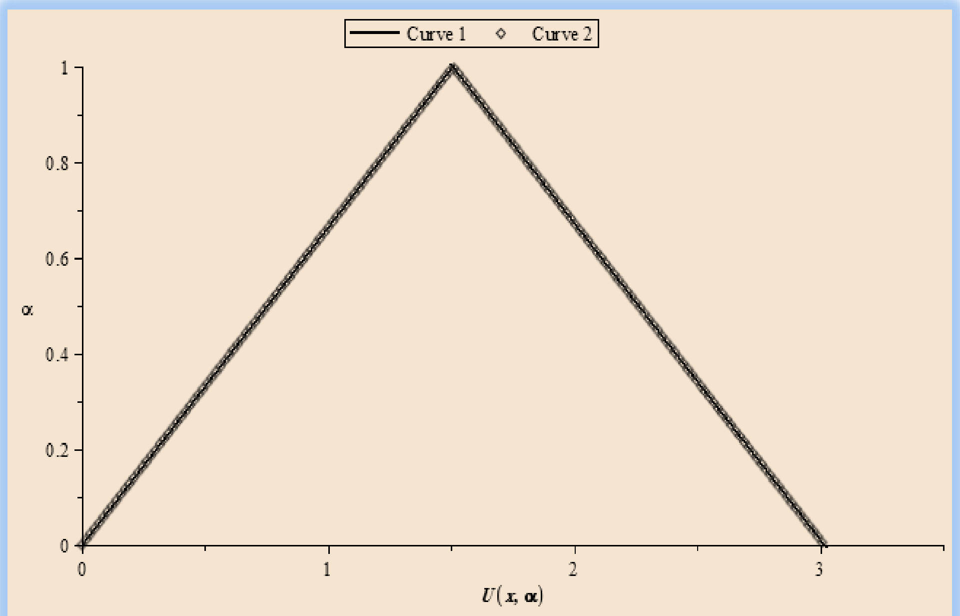
$$\begin{cases} \underline{u}(x,\alpha)=\sum_{r=0}^{s}\left(\frac{1}{r!}\left.\frac{\partial^{(r)}\underline{u}_j(x,\alpha)}{\partial x^r}\right|_{x=z}\left(x-\frac{\pi}{12}\right)^r\right) \\ \overline{u}(x,\alpha)=\sum_{r=0}^{s}\left(\frac{1}{r!}\left.\frac{\partial^{(r)}\overline{u}_j(x,\alpha)}{\partial x^r}\right|_{x=z}\left(x-\frac{\pi}{12}\right)^r\right) \end{cases}$$

$0\leq x\leq 1$, $0\leq\alpha\leq 1$

the approximated solution is:

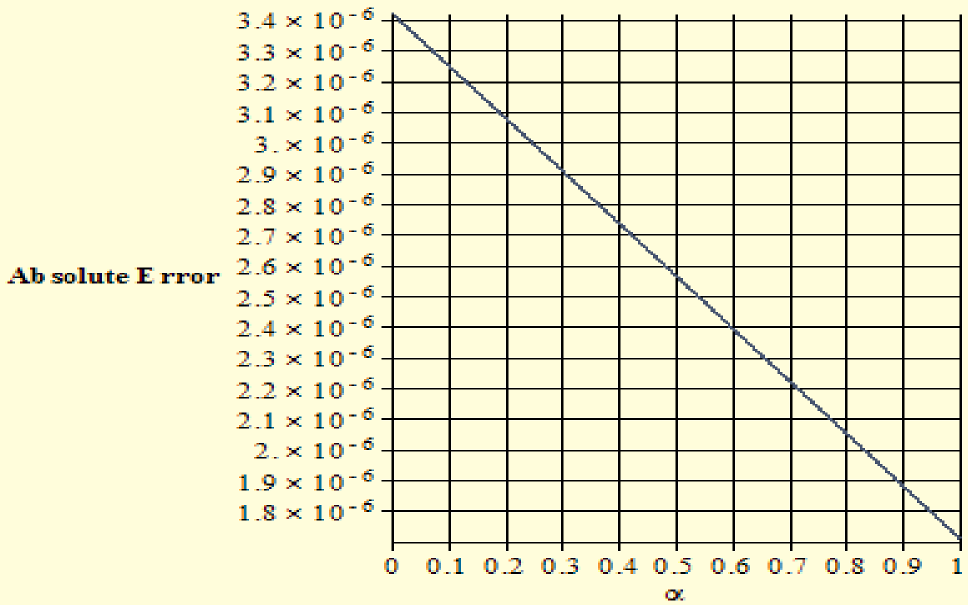
$$\underline{u}(x,\alpha)= 0.9999816737\,\alpha+ 1.000399233\,\alpha\,x + 0.4962378250\,\alpha\,x^2 + 0.5186782119\alpha\,x^3 + 0.2403470047\,\alpha\,x^4 + 0.1803786182\,\alpha\,x^5$$

$\underline{u}(x,\alpha) = 2.000407015\,x + 1.738708618\,\alpha + 0.3524026447\,x^3 + 0.1975243281\,x^4 + 0.1905848255\,x^5 - 0.7387269443 + 0.04282267662\,\alpha\,x^4 - 0.01020620726\,\alpha\,x^5 - 1.000007782\,\alpha\,x + 0.9961625156\,x^2 - 0.4999246903\,\alpha\,x^2 + 0.1662755673\,\alpha\,x^3$



Figure(4.2) (a) the exact and approximated solution at x=$\frac{\pi}{8}$

Figure (4.2) (a) compares between the exact and numerical solutions for $u(x,\alpha)$ in Example (4.2) at $x = \frac{\pi}{8}$



Figure(4.2)(b) absolute error between the exact and numerical solutions at x=$\frac{\pi}{8}$

Figure (4.2) (b) shows the absolute error between the exact and numerical solutions for $U(x, \alpha)$ in Example (4.2) at $x = \frac{\pi}{8}$

Table (4.2) shows a comparison between the exact solution and the approximate solution, and the resulted error.

Table 4.2: The exact and numerical solutions for $u(x,\alpha)$, and the resulted error by algorithm (4.1) $at\ x = \frac{\pi}{8}$

α	Numerical solution $\underline{u}(x,\alpha)$	Exact solution $\underline{u}(x,\alpha)$	Numerical solution $\overline{u}(x,\alpha)$	Exact solution $\overline{u}(x,\alpha)$	Error =D (u_{exact}, u_{approx})
0	0	0	3.016349432	3.01635285	3.4210×10^{-6}
0.1	0.1508174716	0.15081764	2.865531960	2.86553521	3.2500×10^{-6}
0.2	0.3016349432	0.30163528	2.714714489	2.71471756	3.0790×10^{-6}
0.3	0.4524524148	0.45245292	2.563897017	2.56389992	2.9080×10^{-6}
0.4	0.6032698864	0.60327057	2.413079546	2.41308228	2.7360×10^{-6}
0.5	0.7540873580	0.75408821	2.262262074	2.26226464	2.5660×10^{-6}
0.6	0.9049048296	0.90490585	2.111444602	2.11144699	2.3950×10^{-6}
0.7	1.055722301	1.05572349	1.960627131	1.96062935	2.2230×10^{-6}
0.8	1.206539773	1.20654114	1.809809659	1.80981171	2.0520×10^{-6}
0.9	1.357357244	1.35735878	1.658992188	1.65899406	1.8806×10^{-6}
1.0	1.508174716	1.50817642	1.508174716	1.50817642	1.7104×10^{-6}

These results show the accuracy of the Taylor expansion method to solve the equation (4.2) with a maximum error

$$= 3.4210 \times 10^{-6}.$$

Numerical example $(\mathbf{4.3})$: $(\boldsymbol{Trapezoidal\ \ method})$

The fuzzy Volterra integral equations (4.1) have the analytical solution (4.2)

The following algorithm implements the trapezoidal rule using the MAPLE software.

Algorithm $(\mathbf{4.2})$

1. Input $x_0 = a, x_s = b,\ \lambda\ , k_{i,j}(x,t), \underline{f_i}(x,\alpha), \overline{f_i}(x,\alpha)$
2. Input $t_j = x_j, for\ j = 0,1,, \dots, s$
3. Input $(\ s)$ the number of subdivisions of $[a,b]$
4. Calculate $L = \frac{b-a}{s}$, s= $0,1, \dots, p$
5. Calculate $x_{i=}\ i.L \quad i = 0, \dots, s$
6. Put $W_{1,1} = W_{i,j} = \begin{cases} -L.k\left(x_i, x_j\right) & 1 \le j < i \le s \\ 1 - \frac{L}{2}.k(x_i, x_i) & 1 \le i \le s \\ 1 & i = j = 0 \\ -\frac{L}{2}.k(x_i, x_0) & 1 \le i \le s \\ 0 & else\ where \end{cases}$
7. Put $W_{1,1} = W_{2,2}$

8. Put

$$W_{1,2} = W_{2,1} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}_{(s+1)\times(s+1)}$$

9. denote $W = \begin{bmatrix} W_{1,1} & W_{1,2} \\ W_{2,1} & W_{2,2} \end{bmatrix}$,

10. put

$$\mathcal{F} = \begin{bmatrix} -\underline{f_1}(z,\alpha), \dots, -\underline{f_1}^{(k)}(z,\alpha), -\overline{f}_1(z,\alpha), \dots, -\overline{f}_1^{(s)}(z,\alpha), \dots, \\ -\underline{f_m}(z,\alpha), \dots, -\underline{f_m}^{(s)}(z,\alpha), -\overline{f}_m(z,\alpha), \dots, -\overline{f}_m^{(s)}(z,\alpha) \end{bmatrix}^t$$

11. put

$$u = \begin{bmatrix} \underline{u}_{1s}(z,\alpha), \dots, \underline{u}_{1s}^{(s)}(z,\alpha), \overline{u}_{1s}(z,\alpha), \dots, \overline{u}_{1s}^{(s)}(z,\alpha), \dots, \\ \underline{u}_{ms}(z,\alpha), \dots, \underline{u}_{ms}^{(s)}(z,\alpha), \overline{u}_{ms}(z,\alpha), \dots, \overline{u}_{ms}^{(s)}(z,\alpha) \end{bmatrix}^t$$

12. solve the following linear system $\quad Wu = \mathcal{F}$

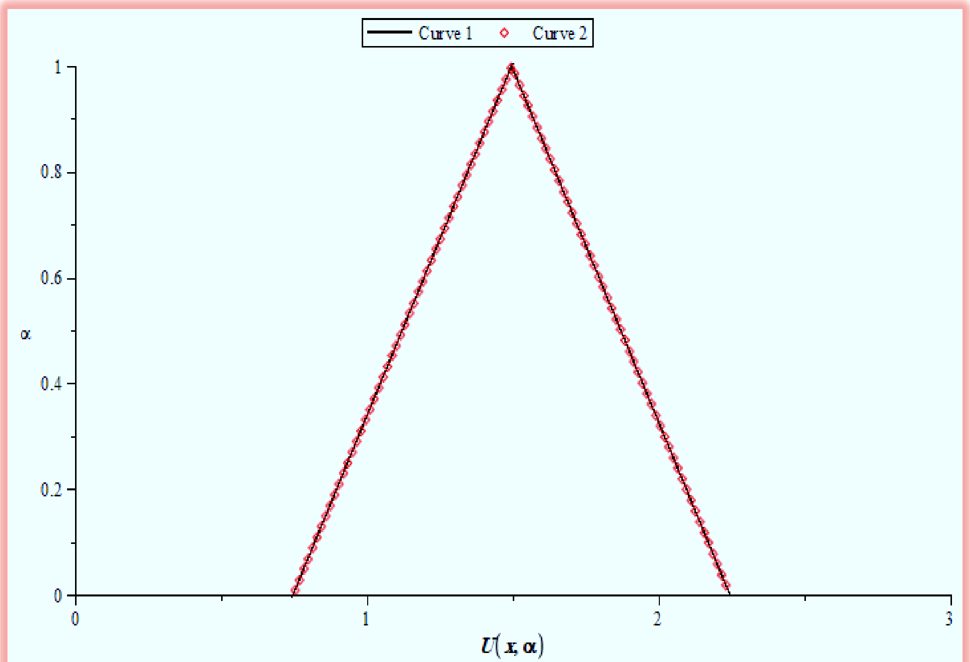
13. estimate $\quad \underline{u}(z,\alpha), \overline{u}(z,\alpha)$

If we take $x = 21.L$, we get :

$$\underline{u}(x,\alpha) = 0.7479719827489786.\alpha + 0.7479719827489785\,,$$

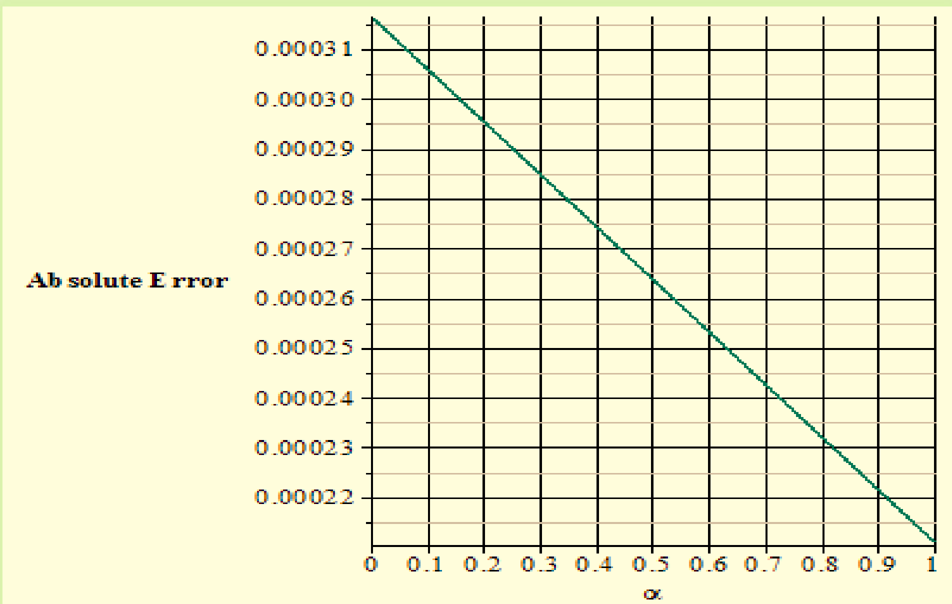
and

$$\overline{u}(x,\alpha) = -0.7479719827489785.\alpha + 2.2439159482469355.$$



Figure(4.3) (a) the exact and approximated solution at x=0.84

Figure (4.3) (a) compares the exact and numerical solutions for $u(x, \alpha)$ in Example (4.3) at $x = 0.84$ with $s = 25$



Figure(4.3)(b) absolute error between the exact and numerical solutions at x=0.84

Figure (4.3) (b) shows the absolute error between the exact and numerical solutions in Example (4.3) at $x = 0.84$ *with* $s = 25$

Table 4.3: The exact and numerical solutions for $u(x,\alpha)$, and the resulted error by algorithm (4.2) $at\ x = 0.84$

α	Numerical solution $\underline{u}(x,\alpha)$	Exact solution $\underline{u}(x,\alpha)$	Numerical solution $\overline{u}(x,\alpha)$	Exact solution $\overline{u}(x,\alpha)$	Error =D (u_{exact}, u_{approx})
0	0.747971982	0.7480775000	2.243915948246	2.244232500	3.1655179×10^{-4}
0.1	0.822769181	0.8228852500	2.169118749972	2.169424750	3.0600002×10^{-4}
0.2	0.897566379	0.8976930000	2.094321551697	2.094617000	2.9544830×10^{-4}
0.3	0.972363577	0.9725007500	2.019524353422	2.019809250	2.8489657×10^{-4}
0.4	1.047160775	1.047308500	1.944727155147	1.945001500	2.7434485×10^{-4}
0.5	1.121957974	1.122116250	1.869929956872	1.870193750	2.6379312×10^{-4}
0.6	1.196755172	1.196924000	1.795132758597	1.870193750	2.5324140×10^{-4}
0.7	1.271552370	1.271731750	1.720335560322	1.720578250	2.4268967×10^{-4}
0.8	1.346349568	1.346539500	1.645538362047	1.645770500	2.3213795×10^{-4}
0.9	1.421146767	1.421347250	1.570741163772	1.570962750	2.2158622×10^{-4}
1.0	1.495943965	1.496155000	1.495943965497	1.496155000	2.1103450×10^{-4}

These results show the efficiency of the Trapezoidal method to solve the equation (4.1) with a maximum error

$$= 3.1655179 \times 10^{-4}$$

Numerical example $(\mathbf{4.4})$: $(\boldsymbol{Trapezoidal\ method})$

The fuzzy Volterra integral equations (4.3) have the analytical solution (4.4)

Algorithm (4.2) implements the trapezoidal method using the MAPLE software, on the interval $\left[0, \frac{\pi}{4}\right]$ with s=25

If we take $x = 23.$ L =0.23π , we get:

$$\underline{u}(x, \alpha) = 2.275392420\,.\,\alpha, \text{ where , } 0 \leq \alpha \leq 1$$

$$\overline{u}(x, \alpha) = -2.275392420.\,\alpha + 4.550784838.$$

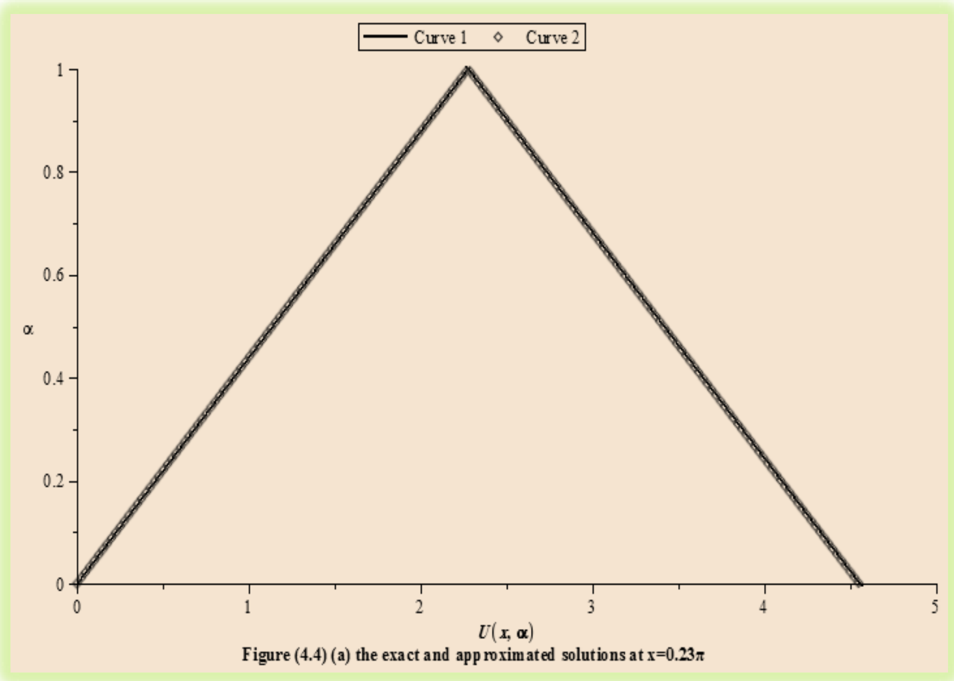


Figure (4.4) (a) compares the exact and numerical solutions for $u(x,\alpha)$ in Example (4.4) at $x = 0.23\pi$ with $s = 25$

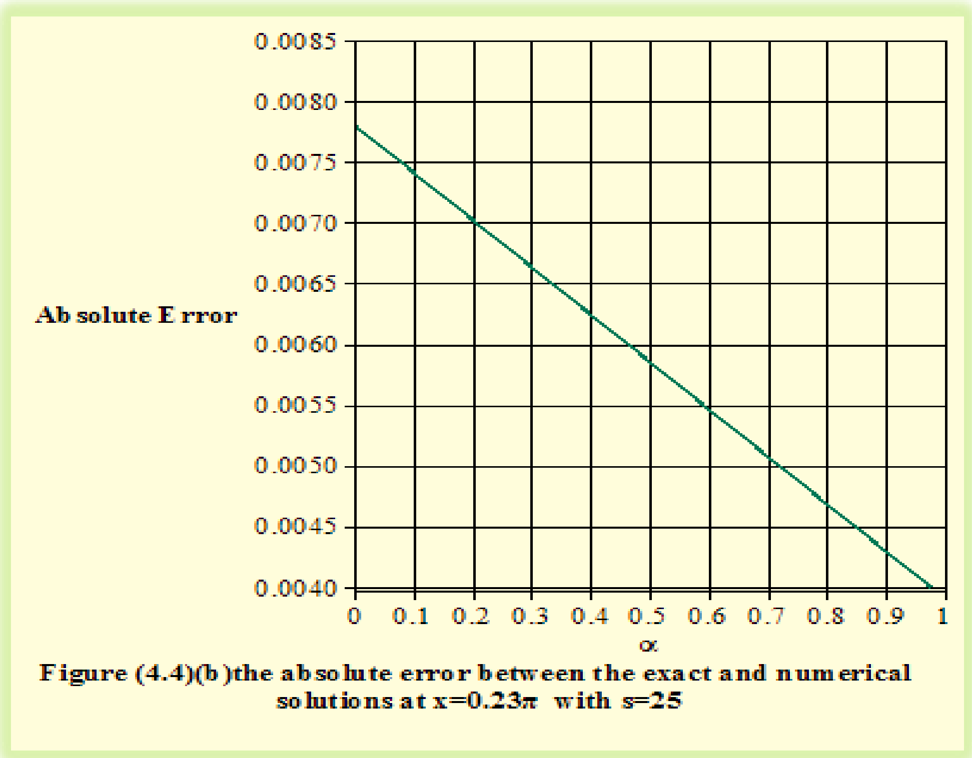


Figure (4.4) (b) shows the absolute error between the exact and numerical solutions for$u(x,\alpha)$ in Example (4.4) at $x = 0.23\pi$ with $s = 25$

Table 4.4: The exact and numerical solutions for $u(x,\alpha)$, and the resulted error by algorithm (4.2) at $x = 0.23\pi$ with $s = 25$

α	Numerical solution $\underline{u}(x,\alpha)$	Exact solution $\underline{u}(x,\alpha)$	Numerical solution $\overline{u}(x,\alpha)$	Exact solution $\overline{u}(x,\alpha)$	Error =D (u_{exact}, U_{appr})
0	**0**	**0**	**4.550784838**	**4.55858982**	**7.804982$\times 10^{-3}$**
0.1	**0.2275392420**	**0.2279294911**	**4.323245596**	**4.33066033**	**7.414733$\times 10^{-3}$**
0.2	**0.4550784840**	**0.4558589821**	**4.095706354**	**4.10273083**	**7.024484$\times 10^{-3}$**
0.3	**0.6826177260**	**0.6837884731**	**3.868167112**	**3.874801348**	**6.634235$\times 10^{-3}$**
0.4	**0.9101569680**	**0.9117179641**	**3.640627870**	**3.64687185**	**6.243986$\times 10^{-3}$**
0.5	**1.137696210**	**1.139647455**	**3.413088628**	**3.41894236**	**5.853736$\times 10^{-3}$**
0.6	**1.365235452**	**1.367576946**	**3.185549386**	**3.19101287**	**5.463487$\times 10^{-3}$**
0.7	**1.592774694**	**1.595506437**	**2.958010144**	**2.96308338**	**5.073238$\times 10^{-3}$**
0.8	**1.820313936**	**1.823435928**	**2.730470902**	**2.73515389**	**4.682989$\times 10^{-3}$**
0.9	**2.047853178**	**2.051365419**	**2.502931660**	**2.50722440**	**4.292740$\times 10^{-3}$**
1.0	**2.275392420**	**2.279294911**	**2.275392418**	**2.279294911**	**3.902491$\times 10^{-3}$**

These results show the efficiency of the Trapezoidal method to solve the equation (4.3) with a maximum error

$$= 7.804982 \times 10^{-3}.$$

We notice from our numerical test cases that the Taylor expansion method is more accurate than the trapezoidal method.

Numerical example $(\mathbf{4.5})$: $(\boldsymbol{variational\ iteration\ method})$

The fuzzy Volterra integral equation(4.1) have the exact solution(4.2).

In the view of the variational iteration method, we construct a correction functional in the following form:

$$\underline{u}_{p+1}(x,\alpha) = \underline{f}(x,\alpha) + \int_0^x k(x,t)\underline{u}_p(t,\alpha)dt,$$

and $$(4.5)$$

$$\overline{u}_{p+1}(x,\alpha) = \overline{f}(x,\alpha) + \int_0^x k(x,t)\overline{u}_p(t,\alpha)dt.$$

where $p = 0,1,2, \dots, s$

Starting with the initial approximation:

$$\underline{u}_0(x,\alpha) = (\alpha + 1)x^2,$$

and

$$\overline{u}_0(x,\alpha) = (3 - \alpha)x^2.$$

in equation (4.1) successive approximations $u_p(x,\alpha)'s$ will be achieved.

In this example we calculate the 8th order of approximate solution using the variational iteration method by MAPLE software.

Algorithm $(\mathbf{4.3})$

1. Input $a, b, \lambda, k(x,t), s, \underline{f}(x,\alpha), \overline{f}(x,\alpha)$
2. Input $\underline{u}_0(x,\alpha), \overline{u}_0(x,\alpha)$
3. for $p = 1\ to\ s+1$, compute

$$\underline{u}_{p+1}(x,\alpha) = \underline{f}(x,\alpha) + \int_0^x k(x,t)\underline{u}_p(t,\alpha)dt,$$

and

$$\overline{u}_{p+1}(x,\alpha) = \overline{f}(x,\alpha) + \int_0^x k(x,t)\overline{u}_p(t,\alpha)dt$$

We obtain the following results:

$$\underline{u}_1(x,\alpha) = (\alpha+1)x^2 - \frac{1}{4}(\alpha+1)x^4 + \frac{1}{3}x^4(\alpha+1)$$

$$\underline{u}_2(x,\alpha) = (\alpha+1)x^2 - \frac{1}{6}\left(\frac{1}{12}\alpha + \frac{1}{12}\right)x^6 + \frac{1}{5}x^6\left(\frac{1}{12}\alpha + \frac{1}{12}\right) - \frac{1}{4}x^4(\alpha+1) + \frac{1}{3}x^4(\alpha+1)$$

$$\underline{u}_3(x,\alpha) = (\alpha+1)x^2 - \frac{1}{8}\left(\frac{1}{360}\alpha + \frac{1}{360}\right)x^8 + \frac{1}{7}x^8\left(\frac{1}{360}\alpha + \frac{1}{360}\right) - \frac{1}{6}x^6\left(\frac{1}{12}\alpha + \frac{1}{12}\right) + \frac{1}{5}x^6\left(\frac{1}{12}\alpha + \frac{1}{12}\right) - \frac{1}{4}x^4(\alpha+1) + \frac{1}{3}x^4(\alpha+1)$$

$$\underline{u}_4(x,\alpha) = (\alpha+1)x^2 - \frac{1}{10}\left(\frac{1}{20160}\alpha + \frac{1}{20160}\right)x^{10} + \frac{1}{9}x^{10}\left(\frac{1}{20160}\alpha + \frac{1}{20160}\right) - \frac{1}{8}\left(\frac{1}{360}\alpha + \frac{1}{360}\right)x^8 + \frac{1}{7}x^8\left(\frac{1}{360}\alpha + \frac{1}{360}\right) - \frac{1}{6}x^6\left(\frac{1}{12}\alpha + \frac{1}{12}\right) + \frac{1}{5}x^6\left(\frac{1}{12}\alpha + \frac{1}{12}\right) - \frac{1}{4}x^4(\alpha+1) + \frac{1}{3}x^4(\alpha+1)$$

$$\underline{u}_5(x,\alpha) = (\alpha+1)x^2 - \frac{1}{12}\left(\frac{1}{1814400}\alpha + \frac{1}{1814400}\right)x^{12} + \frac{1}{11}x^{12}\left(\frac{1}{1814400}\alpha + \frac{1}{1814400}\right) - \frac{1}{10}\left(\frac{1}{20160}\alpha + \frac{1}{20160}\right)x^{10} +$$

$$\frac{1}{9}x^{10}\left(\frac{1}{20160}\alpha+\frac{1}{20160}\right)-\frac{1}{8}\left(\frac{1}{360}\alpha+\frac{1}{360}\right)x^{8}+\frac{1}{7}x^{8}\left(\frac{1}{360}\alpha+\frac{1}{360}\right)-\frac{1}{6}x^{6}\left(\frac{1}{12}\alpha+\frac{1}{12}\right)+\frac{1}{5}x^{6}\left(\frac{1}{12}\alpha+\frac{1}{12}\right)-\frac{1}{4}x^{4}(\alpha+1)+\frac{1}{3}x^{4}(\alpha+1)$$

$$\underline{u}_6(x,\alpha)=\frac{1}{13}\left(\frac{1}{239500800}\alpha+\frac{1}{239500800}\right)x^{14}-\frac{1}{12}\left(\frac{1}{1814400}\alpha+\frac{1}{1814400}\right)x^{12}+\frac{1}{11}x^{12}\left(\frac{1}{1814400}\alpha+\frac{1}{1814400}\right)-\frac{1}{10}\left(\frac{1}{20160}\alpha+\frac{1}{20160}\right)x^{10}+\frac{1}{9}x^{10}\left(\frac{1}{20160}\alpha+\frac{1}{20160}\right)-\frac{1}{8}\left(\frac{1}{360}\alpha+\frac{1}{360}\right)x^{8}+\frac{1}{7}x^{8}\left(\frac{1}{360}\alpha+\frac{1}{360}\right)-\frac{1}{6}x^{6}\left(\frac{1}{12}\alpha+\frac{1}{12}\right)+\frac{1}{5}x^{6}\left(\frac{1}{12}\alpha+\frac{1}{12}\right)-\frac{1}{4}x^{4}(\alpha+1)+\frac{1}{3}x^{4}(\alpha+1)$$

$$\underline{u}_7(x,\alpha)=(\alpha+1)x^2-\frac{1}{16}\left(\frac{1}{43589145600}\alpha+\frac{1}{43589145600}\right)x^{16}+\frac{1}{15}\left(\frac{1}{43589145600}\alpha+\frac{1}{43589145600}\right)x^{16}-\frac{1}{14}\left(\frac{1}{239500800}\alpha+\frac{1}{239500800}\right)x^{14}+\frac{1}{13}\left(\frac{1}{239500800}\alpha+\frac{1}{239500800}\right)x^{14}-\frac{1}{12}\left(\frac{1}{1814400}\alpha+\frac{1}{1814400}\right)x^{12}+\frac{1}{11}x^{12}\left(\frac{1}{1814400}\alpha+\frac{1}{1814400}\right)-\frac{1}{10}\left(\frac{1}{20160}\alpha+\frac{1}{20160}\right)x^{10}+\frac{1}{9}x^{10}\left(\frac{1}{20160}\alpha+\frac{1}{20160}\right)-\frac{1}{8}\left(\frac{1}{360}\alpha+\frac{1}{360}\right)x^{8}+\frac{1}{7}x^{8}\left(\frac{1}{360}\alpha+\frac{1}{360}\right)-\frac{1}{6}x^{6}\left(\frac{1}{12}\alpha+\frac{1}{12}\right)+\frac{1}{5}x^{6}\left(\frac{1}{12}\alpha+\frac{1}{12}\right)-\frac{1}{4}x^{4}(\alpha+1)+\frac{1}{3}x^{4}(\alpha+1)$$

$$\underline{u}_8(x,\alpha)=(\alpha+1)x^2-\frac{1}{18}\left(\frac{1}{10461394944000}\alpha+\frac{1}{10461394944000}\right)x^{18}+\frac{1}{17}\left(\frac{1}{10461394944000}\alpha+\frac{1}{10461394944000}\right)x^{18}+\frac{1}{16}\left(\frac{1}{43589145600}\alpha+\frac{1}{43589145600}\right)x^{16}+\frac{1}{15}\left(\frac{1}{43589145600}\alpha+\frac{1}{43589145600}\right)x^{16}-\frac{1}{14}\left(\frac{1}{239500800}\alpha+\frac{1}{239500800}\right)x^{14}+\frac{1}{13}\left(\frac{1}{239500800}\alpha+\frac{1}{239500800}\right)x^{14}-\frac{1}{12}\left(\frac{1}{1814400}\alpha+\frac{1}{1814400}\right)x^{12}+\frac{1}{11}x^{12}\left(\frac{1}{1814400}\alpha+\frac{1}{1814400}\right)-\frac{1}{10}\left(\frac{1}{20160}\alpha+\frac{1}{20160}\right)x^{10}+\frac{1}{9}x^{10}\left(\frac{1}{20160}\alpha+\frac{1}{20160}\right)-\frac{1}{8}\left(\frac{1}{360}\alpha+\frac{1}{360}\right)x^{8}+\frac{1}{7}x^{8}\left(\frac{1}{360}\alpha+\frac{1}{360}\right)-\frac{1}{6}x^{6}\left(\frac{1}{12}\alpha+\frac{1}{12}\right)+\frac{1}{5}x^{6}\left(\frac{1}{12}\alpha+\frac{1}{12}\right)-\frac{1}{4}x^{4}(\alpha+1)+\frac{1}{3}x^{4}(\alpha+1)$$

and

$$\overline{u}_1(x,\alpha) = (\alpha+1)x^2 - \frac{1}{4}(3-\alpha)x^4 + \frac{1}{3}x^4(3-\alpha)$$

$$\overline{u}_2(x,\alpha) = (3-\alpha)x^2 - \frac{1}{6}\left(\frac{1}{4} - \frac{1}{12}\alpha\right)x^6 + \frac{1}{5}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) - \frac{1}{4}x^4(3-\alpha) + \frac{1}{3}x^4(3-\alpha)$$

$$\overline{u}_3(x,\alpha) = (3-\alpha)x^2 - \frac{1}{8}\left(\frac{1}{120} - \frac{1}{360}\alpha\right)x^8 + \frac{1}{7}x^8\left(\frac{1}{120} - \frac{1}{360}\alpha\right) - \frac{1}{6}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) + \frac{1}{5}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) - \frac{1}{4}x^4(3-\alpha) + \frac{1}{3}x^4(3-\alpha)$$

$$\overline{u}_4(x,\alpha) = (3-\alpha)x^2 - \frac{1}{10}\left(\frac{1}{6720} - \frac{1}{20160}\alpha\right)x^{10} + \frac{1}{9}x^{10}\left(\frac{1}{6720} - \frac{1}{20160}\alpha\right) - \frac{1}{8}\left(\frac{1}{120} - \frac{1}{360}\alpha\right)x^8 + \frac{1}{7}x^8\left(\frac{1}{120} - \frac{1}{360}\alpha\right) - \frac{1}{6}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) + \frac{1}{5}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) - \frac{1}{4}x^4(3-\alpha) + \frac{1}{3}x^4(3-\alpha)$$

$$\overline{u}_5(x,\alpha) = (3-\alpha)x^2 - \frac{1}{12}\left(\frac{1}{604800} - \frac{1}{1814400}\alpha\right)x^{12} + \frac{1}{11}x^{12}\left(\frac{1}{604800} - \frac{1}{1814400}\alpha\right) - \frac{1}{10}\left(\frac{1}{6720} - \frac{1}{20160}\alpha\right)x^{10} + \frac{1}{9}x^{10}\left(\frac{1}{6720} - \frac{1}{20160}\alpha\right) - \frac{1}{8}\left(\frac{1}{120} - \frac{1}{360}\alpha\right)x^8 + \frac{1}{7}x^8\left(\frac{1}{120} - \frac{1}{360}\alpha\right) - \frac{1}{6}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) + \frac{1}{5}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) - \frac{1}{4}x^4(3-\alpha) + \frac{1}{3}x^4(3-\alpha).$$

$$\overline{u}_6(x,\alpha) = (3-\alpha)x^2 - \frac{1}{14}\left(\frac{1}{79833600} - \frac{1}{239500800}\alpha\right)x^{14} + \frac{1}{13}\left(\frac{1}{79833600} - \frac{1}{239500800}\alpha\right)x^{14} - \frac{1}{12}\left(\frac{1}{604800} - \frac{1}{1814400}\alpha\right)x^{12} + \frac{1}{11}x^{12}\left(\frac{1}{604800} - \frac{1}{1814400}\alpha\right) - \frac{1}{10}\left(\frac{1}{6720} - \frac{1}{20160}\alpha\right)x^{10} + \frac{1}{9}x^{10}\left(\frac{1}{6720} - \frac{1}{20160}\alpha\right) - \frac{1}{8}\left(\frac{1}{120} - \frac{1}{360}\alpha\right)x^8 + \frac{1}{7}x^8\left(\frac{1}{120} - \frac{1}{360}\alpha\right) - \frac{1}{6}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) + \frac{1}{5}x^6\left(\frac{1}{4} - \frac{1}{12}\alpha\right) - \frac{1}{4}x^4(3-\alpha) + \frac{1}{3}x^4(3-\alpha).$$

$$\overline{u}_7(x,\alpha) = (3-\alpha)x^2 - \frac{1}{16}\left(\frac{1}{14529715200} - \frac{1}{43589145600}\alpha\right)x^{16} + \frac{1}{15}\left(\frac{1}{14529715200} - \frac{1}{43589145600}\alpha\right)x^{16} - \frac{1}{14}\left(\frac{1}{79833600} - \frac{1}{239500800}\alpha\right)x^{14} + \frac{1}{13}\left(\frac{1}{79833600} - \frac{1}{239500800}\alpha\right)x^{14} -$$

$$\frac{1}{12}\left(\frac{1}{604800}-\frac{1}{1814400}\alpha\right)x^{12}+\frac{1}{11}x^{12}\left(\frac{1}{604800}-\frac{1}{1814400}\alpha\right)-\frac{1}{10}\left(\frac{1}{6720}-\frac{1}{20160}\alpha\right)x^{10}+\frac{1}{9}x^{10}\left(\frac{1}{6720}-\frac{1}{20160}\alpha\right)-\frac{1}{8}\left(\frac{1}{120}-\frac{1}{360}\alpha\right)x^{8}+\frac{1}{7}x^{8}\left(\frac{1}{120}-\frac{1}{360}\alpha\right)-\frac{1}{6}x^{6}\left(\frac{1}{4}-\frac{1}{12}\alpha\right)+\frac{1}{5}x^{6}\left(\frac{1}{4}-\frac{1}{12}\alpha\right)-\frac{1}{4}x^{4}(3-\alpha)+\frac{1}{3}x^{4}(3-\alpha).$$

$$\overline{u}_8(x,\alpha)=(3-\alpha)x^2 \quad -\frac{1}{18}\left(\frac{1}{3487131648000}-\frac{1}{10461394944000}\alpha\right)x^{18}+\frac{1}{17}\left(\frac{1}{3487131648000}-\frac{1}{10461394944000}\alpha\right)x^{18}-\frac{1}{16}\left(\frac{1}{14529715200}-\frac{1}{43589145600}\alpha\right)x^{16}+\frac{1}{15}\left(\frac{1}{14529715200}-\frac{1}{43589145600}\alpha\right)x^{16}-\frac{1}{14}\left(\frac{1}{79833600}-\frac{1}{239500800}\alpha\right)x^{14}+\frac{1}{13}\left(\frac{1}{79833600}-\frac{1}{239500}\alpha\right)x^{14}-\frac{1}{12}\left(\frac{1}{604800}-\frac{1}{1814400}\alpha\right)x^{12}+\frac{1}{11}x^{12}\left(\frac{1}{604800}-\frac{1}{1814400}\alpha\right)-\frac{1}{10}\left(\frac{1}{6720}-\frac{1}{20160}\alpha\right)x^{10}+\frac{1}{9}x^{10}\left(\frac{1}{6720}-\frac{1}{20160}\alpha\right)-\frac{1}{8}\left(\frac{1}{120}-\frac{1}{360}\alpha\right)x^{8}+\frac{1}{7}x^{8}\left(\frac{1}{120}-\frac{1}{360}\alpha\right)-\frac{1}{6}x^{6}\left(\frac{1}{4}-\frac{1}{12}\alpha\right)+\frac{1}{5}x^{6}\left(\frac{1}{4}-\frac{1}{12}\alpha\right)-\frac{1}{4}x^{4}(3-\alpha)+\frac{1}{3}x^{4}(3-\alpha)\ .$$

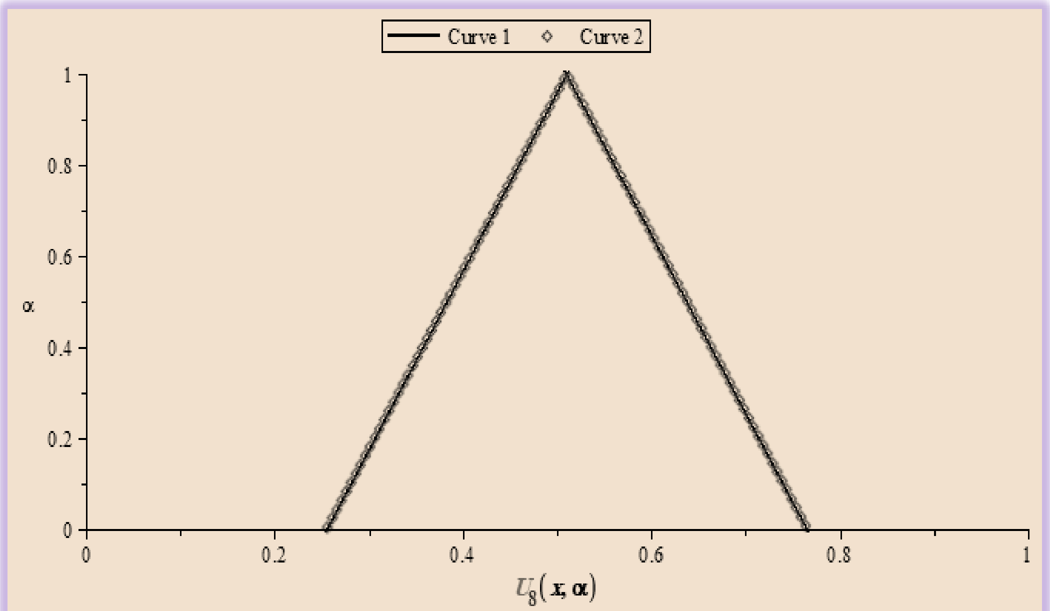


Figure (4.5) (a) the exact and approximate solution at x= 0.5

Figure (4.5) **(a)** compares the exact and numerical solutions for $u(x,\alpha)$ in Example (4.5) at $x=0.5$

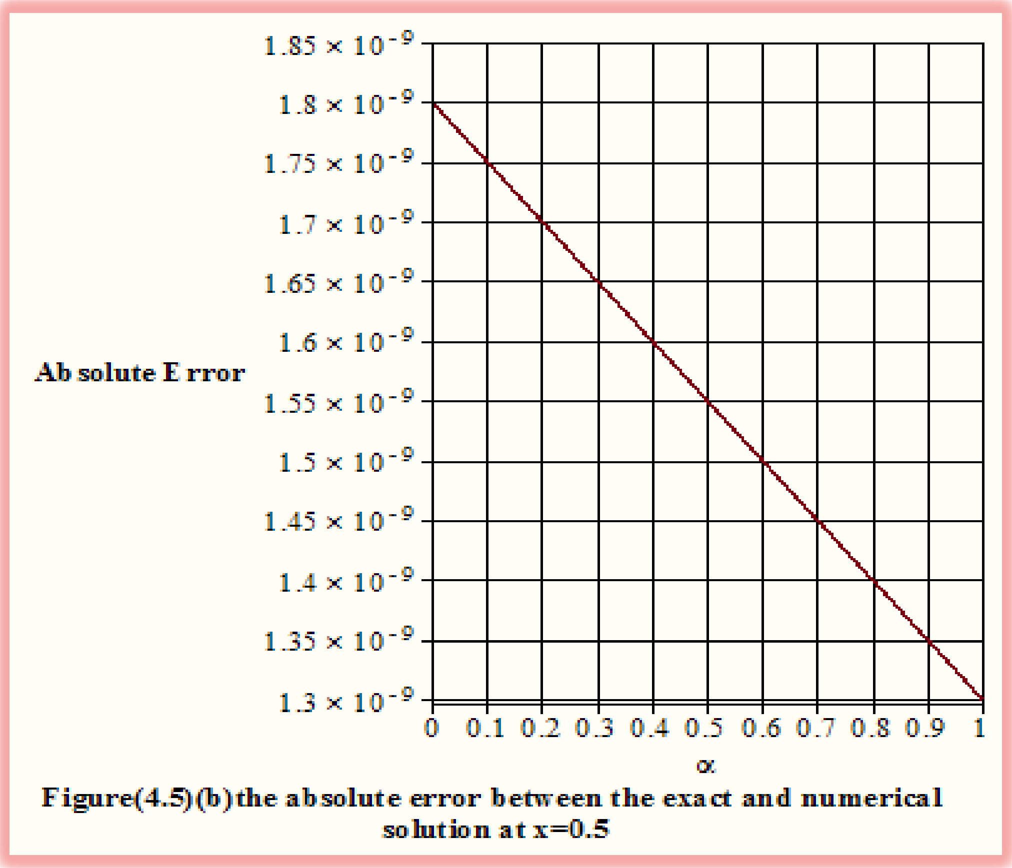


Figure (4.5) (b) shows the absolute error between the exact and numerical solutions for $u(x,\alpha)$ in Example (4.5) at $x = 0.5$

Table $(4.5)(a)$ shows the exact and numerical results when applying algorithm(4.3), and showing the absolute error between the exact and numerical solutions of equation(4.1)

Table 4.5: the exact and numerical results, the absolute resulted error of applying algorithm (4.3) for equation(4.1) $at\ x = 0.5$.

α	Numerical solution $\underline{u}(x,\alpha)$	Exact solution $\underline{u}(x,\alpha)$	Numerical solution $\overline{u}(x,\alpha)$	Exact solution $\overline{u}(x,\alpha)$	Error =D (u_{exact}, u_{approx})
0	0.2552519305	0.255251931	0.7657557912	0.765755793	1.8000×10^{-9}
0.1	0.2807771236	0.2807771241	0.7402305982	0.7402305999	1.7500×10^{-9}
0.2	0.3063023166	0.3063023172	0.7147054051	0.7147054068	1.7000×10^{-9}
0.3	0.3318275096	0.3318275103	0.6891802120	0.6891802137	1.6500×10^{-9}
0.4	0.3573527027	0.3573527034	0.6636550190	0.6636550206	1.6000×10^{-9}
0.5	0.3828778957	0.3828778965	0.6381298260	0.6381298275	1.5500×10^{-9}
0.6	0.4084030888	0.4084030896	0.6126046329	0.6126046344	1.5000×10^{-9}
0.7	0.4339282819	0.4339282827	0.5870794398	0.5870794413	1.4500×10^{-9}
0.8	0.4594534749	0.4594534758	0.5615542468	0.5615542482	1.4000×10^{-9}
0.9	0.4849786679	0.4849786689	0.5360290538	0.5360290551	1.3500×10^{-9}
1.0	0.5105038610	0.5105038620	0.5105038607	0.5105038620	1.3000×10^{-9}

The results showed that the convergence and accuracy of variational iteration method for numerical solution the Volterra integral equations were in a good agreement with the analytical solutions.

Error $=D\left(u_{exact}(.5,\alpha), u_{approximate}(.5,\alpha)\right)$

$$= \sup_{0\le\alpha\le1}\{max|\underline{u}_{exact}-\underline{u}_{approximate}|, max|\underline{u}_{exact}-\underline{u}_{approximate}|\}$$

$$=1.8\times 10^{-9}$$

Numerical example $(\mathbf{4.6})$: ($\boldsymbol{variational\ iteration\ method}$)

The fuzzy Volterra integral equation(4.3) have the exact solution(4.4).

In the view of the variational iteration method, we construct a correction functional in the following form:

$$\underline{u}_{p+1}(x,\alpha)=\underline{f}(x,\alpha)+\int_0^x k(x,t)\underline{u}_p(t,\alpha)dt,$$

and

$$\overline{u}_{p+1}(x,\alpha)=\overline{f}(x,\alpha)+\int_0^x k(x,t)\overline{u}_p(t,\alpha)dt.$$

where $p=0,1,2,\dots,s$

Starting with the initial approximation:

$$\underline{u}_0(x,\alpha)=(\alpha)\cos x,$$

and

$$\overline{u}_0(x,\alpha)=(2-\alpha)\cos x.$$

where , $0 \leq \alpha \leq 1$

In equation (4.3) successive approximations $u_p(x,\alpha)'s$ will be achieved.

In this example we calculate the 6th order of approximate solution using the variational iteration method by MAPLE software.

applying algorithm(4.3), we obtain the following results:

$$\underline{u}_1(x,\alpha) = \frac{1}{2}\alpha\cos x + \frac{1}{2}\alpha e^x + \frac{1}{2}\alpha\sin x$$

$$\underline{u}_2(x,\alpha) = \frac{1}{2}\alpha\cos x + \frac{1}{2}\alpha e^x + \frac{1}{2}\alpha e^x x$$

$$\underline{u}_3(x,\alpha) = \frac{3}{4}\alpha\cos x + \frac{1}{4}\alpha e^x + \frac{1}{4}\alpha\sin x + \frac{1}{2}\alpha e^x x + \frac{1}{2}\alpha e^x x^2$$

$$\underline{u}_4(x,\alpha) = \frac{1}{2}\alpha\cos x + \frac{1}{2}\alpha e^x + \frac{1}{4}\alpha\sin x{+}+ \frac{1}{4}\alpha e^x x + \frac{1}{4}\alpha e^x x^2 + \frac{1}{12}\alpha e^x x^3$$

$$\underline{u}_5(x,\alpha) = \frac{5}{8}\alpha\cos x + \frac{3}{8}\alpha e^x + \frac{1}{8}\alpha\sin x{+}+ \frac{1}{2}\alpha e^x x + \frac{1}{8}\alpha e^x x^2 + \frac{1}{12}\alpha e^x x^3 + \frac{1}{48}\alpha e^x x^4$$

$$\underline{u}_6(x,\alpha) = \frac{5}{8}\alpha\cos x + \frac{3}{8}\alpha e^x + \frac{1}{4}\alpha\sin x{+}+ \frac{3}{8}\alpha e^x x + \frac{1}{4}\alpha e^x x^2 + \frac{1}{24}\alpha e^x x^3 + \frac{1}{48}\alpha e^x x^4 \frac{1}{240}\alpha e^x x^5$$

and

$$\overline{u}_1(x,\alpha) = (2-\alpha)\cos x + e^x - \frac{1}{2}\alpha e^x - \cos x + \sin x + \frac{1}{2}\alpha\cos x - \frac{1}{2}\alpha\sin x$$

$\overline{u}_2(x,\alpha) = (2-\alpha)\cos x + e^x - \frac{1}{2}\alpha e^x - \cos x + \frac{1}{2}\alpha\cos x + e^x x - \frac{1}{2}\alpha e^x x.$

$\overline{u}_3(x,\alpha) = (2-\alpha)\cos x + \frac{1}{2}e^x - \frac{1}{4}\alpha e^x - \frac{1}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{4}\alpha\cos x\, e - \frac{1}{4}\alpha\sin x + e^x x - \frac{1}{2}\alpha e^x x + \frac{1}{2}e^x x^2 - \frac{1}{4}\alpha e^x x^2$

$\overline{u}_4(x,\alpha) = (2-\alpha)\cos x + e^x - \frac{1}{2}\alpha e^x - \frac{1}{12}\alpha e^x x^3 + \frac{1}{2}e^x x + \frac{1}{2}e^x x^2 + \frac{1}{6}e^x x^3 - \cos x + \frac{1}{2}\sin x - \frac{1}{4}\alpha e^x x - \frac{1}{4}\alpha e^x x^2 + \frac{1}{2}\alpha\cos x - \frac{1}{4}\alpha\sin x$

$\overline{u}_5(x,\alpha) = (2-\alpha)\cos x + \frac{3}{4}e^x - \frac{3}{8}\alpha e^x + \frac{1}{4}e^x x^2 + \frac{3}{8}\alpha\cos x + \frac{1}{6}e^x x^3 + \frac{1}{4}\sin x + e^x x - \frac{3}{4}\cos x - \frac{1}{48}\alpha e^x x^4 - \frac{1}{8}\alpha\sin x - \frac{1}{12}\alpha e^x x^3 - \frac{1}{8}\alpha e^x x^2 - \frac{1}{2}\alpha e^x x + \frac{1}{24}e^x x^4$

$\overline{u}_6(x,\alpha) = (2-\alpha)\cos x + \frac{3}{4}e^x - \frac{3}{8}\alpha e^x + \frac{1}{2}e^x x^2 + \frac{3}{8}\alpha\cos x + \frac{1}{12}e^x x^3 + \frac{1}{2}\sin x + \frac{3}{4}e^x x - \frac{3}{4}\cos x + \frac{1}{120}e^x x^5 - \frac{1}{240}\alpha e^x x^5 - \frac{1}{48}\alpha e^x x^4 - \frac{1}{4}\alpha\sin x - \frac{1}{24}\alpha e^x x^3 - \frac{1}{4}\alpha e^x x^2 - \frac{3}{8}\alpha e^x x + \frac{1}{24}e^x x^4$

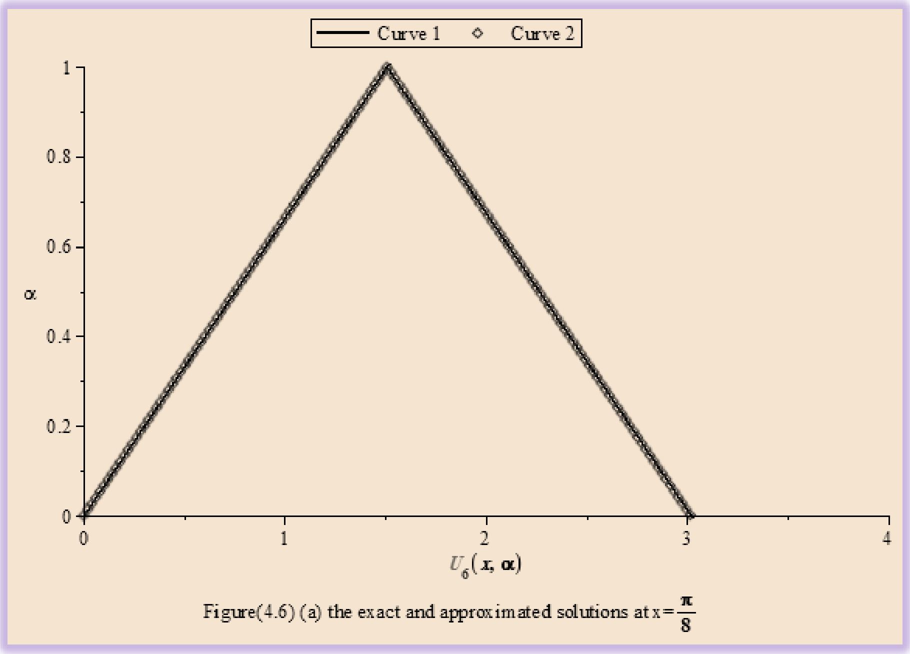


Figure (4.6) (a) compares the exact and numerical solutions for $u(x, \alpha)$ in Example (4.6) at $x = \frac{\pi}{8}$

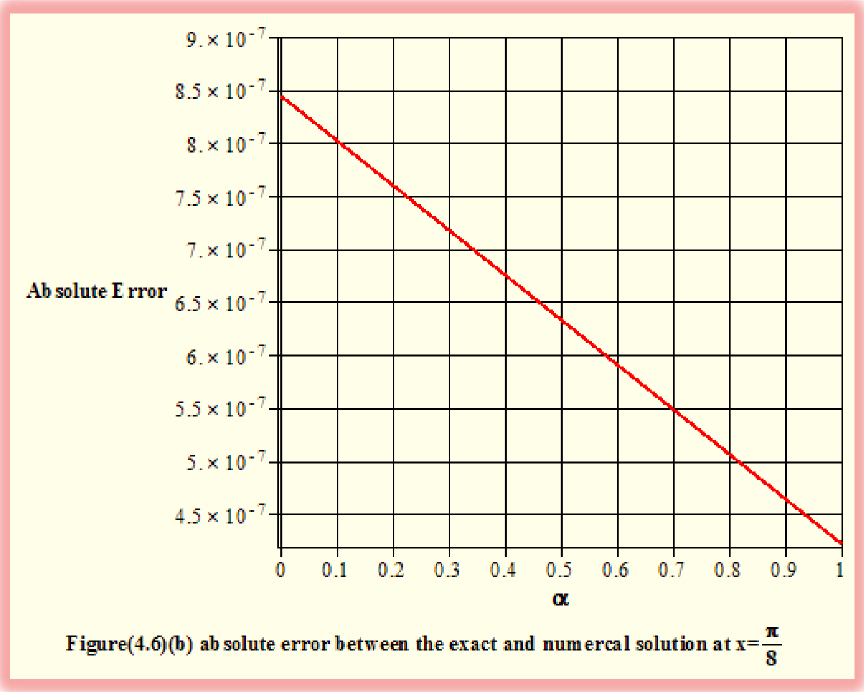


Figure (4.6) (b) shows absolute error between the exact and numerical solutions for $u(x, \alpha)$ in Example (4.6) at $x = \frac{\pi}{8}$

Table (4.6) shows the exact and numerical results when applying algorithm (4.3), and showing the resulted error of equation(4.3).

Table 4.6: the exact and numerical results, the resulted absolute error by applying algorithm (4.3) for equation (4.3) $at\ x = \frac{\pi}{8}$.

α	Numerical solution $\underline{u}(x,\alpha)$	Exact solution $\underline{u}(x,\alpha)$	Numerical solution $\overline{u}(x,\alpha)$	Exact solution $\overline{u}(x,\alpha)$	Error =D (u_{exact}, u_{approx})
0	0	0	3.016352007	3.01635285	$.4600 \times 10^{-7}8$
0.1	0.1508176002	0.1508176426	2.865534406	2.865535210	8.0400$\times 10^{-7}$
0.2	0.3016352007	0.3016352853	2.714716807	2.71471756	7.6100$\times 10^{-7}$
0.3	0.4524528010	0.4524529279	2.563899206	2.56389992	7.1900$\times 10^{-7}$
0.4	0.6032704013	0.6032705706	2.413081606	2.41308228	6.7700$\times 10^{-7}$
0.5	0.7540880017	0.7540882132	2.262264004	2.262264640	6.3600$\times 10^{-7}$
0.6	0.9049056021	0.9049058558	2.111446404	2.11144699	5.9300$\times 10^{-7}$
0.7	1.055723203	1.055723498	1.960628805	1.96062935	5.5000$\times 10^{-7}$
0.8	1.206540802	1.206541141	1.809811204	1.80981171	5.0700$\times 10^{-7}$
0.9	1.357358403	1.357358784	1.658993604	1.65899406	4.6500×10^{-7}
1.0	1.508176002	1.508176426	1.508176002	1.508176426	4.2400×10^{-7}

Table (4.6) shows the convergence and accuracy of variational iteration method for numerical solution the Volterra integral equations were in a good agreement with the analytical solutions , with a maximum error $= 8.46 \times 10^{-7}$

from our numerical test cases, we conclude that the variational iteration method is more efficient than the trapezoidal and the Taylor methods .

Conclusion

In this thesis we have solved the linear fuzzy Volterra integral equation of the second kind using various analytical and numerical methods.

The numerical methods methods were implemented in a form of algorithms to solve some numerical tests cases using MAPLE software

We have obtained the following results

Numerical method	**Maximum error**
Taylor expansion method	3.4210×10^{-6}
Trapezoidal rule	7.804982×10^{-3}
Variation iteration method	8.46×10^{-7}

Numerical results have shown to be in a close agreement with the analytical ones. Moreover, the variation iteration method is one of the most powerful numerical technique for solving fuzzy Volterra integral equation of the second kind in comparison with other numerical techniques.

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كلية الدراسات العليا

حل معادلة فولتيرا التكاملية الخطية الضبابية من النوع الثاني بالطرق التحليلية والعددية

إعداد

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إشراف

أ.د. ناجي قطناني

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات بكلية الدراسات العليا في جامعة النجاح الوطنية، نابلس- فلسطين .

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الملخص

المعادلات التكاملية بشكل عام تلعب دورا هاما جدا في الهندسة والتكنولوجيا لما لها من تطبيقات واسعة. معادلات فولتيرا التكاملية الضبابية بشكل خاص لها العديد من التطبيقات مثل التحكم الضبابية والتحويل والنظم الاقتصادية الضبابية.

بعد أن تناولنا المفاهيم الأساسية في الرياضيات الضبابية، قمنا بالتركيز على الطرق التحليلية والعددية لحل معادلة فولتيرا التكاملية الضبابية من النوع الثاني. ولحل معادلة فولتيرا التكاملية بالطرق التحليلية قدمنا الطرق التالية: طريقة تحويل لابلاس الضبابية، طريقة هوموتوبي التحليلية الضبابية، طريقة أدوميان التحليلية الضبابية، طريقة التحويل التفاضلية الضبابية، طريقة التقريب المتتالي الضبابية.

ولحل معادلة فولتيرا التكاملية بالطرق العددية قمنا بتنفذ طرق مختلفة وهي: طريقة تايلر التوسعية، طريقة شبه المنحرف، طريقة تباين التكرار.

وللتحقق من كفاءة هذه الطرق العددية قمنا بحل بعض الأمثلة العددية، حيث أظهرت النتائج العددية دقتها وقربها من النتائج التحليلية، وكانت طريقة تباين التكرار هي الأقوى والأدق في حل معادلة فولتيرا التكاملية الضبابية من النوع الثاني بالمقارنة مع الطرق العددية الأخرى.