An-Najah National University

Faculty of Graduate Studies

Study of Zariski Topology of Modules, between Theory and Practice

By

Naheel Saber Mohammed Ghaith

Supervisor

Dr. Khalid Adarbeh

This Thesis is Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Mathematics, Faculty of Graduate Studies, An-Najah National University, Nablus-Palestine.

2021

Study of Zariski Topology of Modules, between Theory and Practice

By

Naheel Saber Mohammed Ghaith

This Thesis was defended successfully on 29 /7/2021 and approved by:

Defense Committee Members	**Signature**
– Dr. Khalid Adarbeh/ Supervisor	....
– Dr. Iyad Alhribat/ External Examiner	....
– Dr. Mohammed Abu Eideh/ Internal Examiner	....

Dedication

This thesis is dedicated to my parents, my sisters and brother for their support, as well as to my whole family and friends.

With respect and love.

Acknowledgements

First, I want to thank Allah for giving me the strength and courage to complete this work. I am grateful to my supervisor Dr. Khalid Adarbeh for his continuous help and guidance. Also I want to thank Dr. Mohammed Abu Eideh and Dr. Iyad Alhribat for their helpful guidance and efforts. Finally, I would like to thank all the stuff member of Mathematics Department at An-najah National University for their contribution during my studies.

الإقرار

أنا الموقعة أدناه مقدمة الرسالة التي تحمل العنوان:

Study of Zariski Topology of Modules, between Theory and Practice

أقر بأن ما اشتملت عليه هذه الرسالة إنما هي نتاج جهدي الخاص، باستثناء ما تمت الإشارة إليه حيثما ورد، وأن هذا الرسالة ككل أو أي جزء منها لم يقدم من قبل لنيل أي درجة علمية أو بحث علمي لدى أي مؤسسة تعليمية أو بحثية أخرى.

Declaration

The work provided in this thesis, unless otherwise referenced, is the researcher's own work, and has not been submitted elsewhere for any degree or qualification.

Student's Name: **اسم الطالبة: نهيل صابر محمد غيث**

Signature: **التوقيع:** نهيل غيث

Date: **التاريخ:2021/7/29**

Table of Contents

Dedication III

Acknowledgments... IV

Declaration V

Table of Contents VI

Abstract... VIII

Introduction 1

Preliminaries... 5

0.1 Ring and Module theory 5

0.1.1 Generated ideals 5

0.1.2 Prime ideals and Maximal ldeals 5

0.1.3 Radicals 7

0.1.4 Dimension of a ring 7

0.1.5 Noetherian and Artinian rings... 8

0.1.6 Localization 9

0.1.7 Rings homomorphism 10

0.1.8 Modules... 11

0.1.9 Finitely generated modules 12

0.1.10 Noetherian and Artinian modules 12

0.2 General Topology 13

0.2.1 Separation Axioms 14

Chapter 1 Zariski topology of rings 18

1.1 Definition and Examples... 18

1.2 Properties of the Zariski topology of a ring 23

1.3 Homeomorphism spectrums……………………..…………………..30

1.4 The spectrum of product ring ..……………………………….….…33

Chapter 2 Zariski topology of modules ..…………………………….….. 37

2.1 Definitions and Basics ..…………………………………...….. 37

2.2 Properties of classical Zariski topology ……………………...…… 41

2.3 Classical Zariski topology and Hausdorff spaces ……………….…..46

2.4 Irreducible subsests in classical Zariski topology of modules 51

2.5 Classical Zariski topology in view of spectral spaces ..…………..57

References ………………………………………………………… 67

Study of Zariski Topology of Modules, between Theory and Practice

By
Naheel Ghaith
Supervisor
Dr. Khalid Adarbeh

Abstract

This thesis contributes to study the Zariski topology of rings as well as it's generalization to modules. The first part of the thesis introduce the Zariski topology of rings, several topological properties are discussed, for example compactness, separation axioms, Noetherianity of spaces.

In the second part of the thesis, we study a generalization of Zariski's topology from ring theory into module theory. This generalization was introduced by M.

Behboodi and M. R. Haddadi in 2008. As in the first part, many features of the topologies are presented as a generalization of similar properties in part one.

The theme throught the two parts is how are topological properties related to algebraic properties. It should be noted that the results of the two parts are used to build illustrative examples of topologies.

Introduction

Throughout, $\mathfrak{R}$ denotes a commutative ring with identity element $1_{\mathfrak{R}}$. The spectrum of $\mathfrak{R}$, $\mathrm{Spec}(\mathfrak{R})$, denotes the set of all prime ideals of $\mathfrak{R}$. For a subset $\mathcal{S}$ of $\mathfrak{R}$, $V(\mathcal{S})$ denotes the set of all prime ideals of $\mathfrak{R}$ which contain $\mathcal{S}$.

Let K be a field. The set of all n-tuples $(a_1, a_2, \cdots, a_n)$ of numbers in K is called *affine n-space* and denoted by $\mathbb{A}^n_K$. The ring of polynomials over the field K, denoted by $K[x_1, x_2, \cdots, x_n]$, let $f \in K[x_1, x_2, \cdots, x_n]$ and $a = (a_1, a_2, \cdots, a_n) \in \mathbb{A}^n_K$, then $f(a) \in K$ and so the polynomials are also functions on $\mathbb{A}^n_K$. An *affine variaty* is the set of common zeros of a collection of polynomials. If $S \subset K[x_1, x_2, \cdots, x_n]$, the variaty of S is $V(S) = \{a \in \mathbb{A}^n_K \mid f(a) = 0 \text{ for all } f \in S\}$. Note that, $\emptyset = V(1)$ and $\mathbb{A}^n_K = V(0)$ are variaties. Moreover, the intersection of any collection of affine variaties is an affine variaty and the union of any finite collection of affine variaties is an affine variaty. Hence, the affine variaties have the same properties as the closed topology on $\mathbb{A}^n_K$. This topology was observed by Oscar Zariski in 1952 and it is called the Zariski topology on $\mathbb{A}^n_K$. For any $V \subseteq \mathbb{A}^n_K$, $I(V)$ is the ideal of all polynomial functions vanishing on V, $I(V) = \{f \in K[x_1, x_2, \cdots, x_n] \mid f(x) = 0 \text{ for all } x \in v\}$, the coordinate ring of V is the quotient of the polynomial ring by this ideal. In 1960, Grothendiek transferred the topology to the spectrum of an arbitrary commutative ring by using the correspondence between points of an affine variety and maximal ideals of its coordinate ring. and he called it Zariski topology, for more details see [16, Notes on chapter V].

The *Zariski topology of rings* is a topology defined over $\mathrm{Spec}(\mathfrak{R})$ where the closed sets are of the form $V(\mathcal{S})$, $\mathcal{S}$ a is an arbitrary subset of $\mathfrak{R}$. For $x \in \mathfrak{R}$, let $W(x) := \mathrm{Spec}(\mathfrak{R}) \setminus V(x)$. The collection $\{W(x)|x \in \mathfrak{R}\}$ forms a base of open sets for the Zariski topology of rings. For the basic algebraic, topological and algebraic geometry definitions, we refer the reader to [3, 17, 29, 4].

Chapter one is devoted to study the Zariski topology of rings. We will discuss some basic topological properties of $\mathrm{Spec}(\mathfrak{R})$ and its relation with the algebraic properties of $\mathfrak{R}$ by solving the Zariski topology exercises of Atiyah and MacDonald [3]. Those exercises will form the body of the Zariski topology of rings. The following are some discussed properties of this topology:

1. $\mathrm{Spec}(\mathfrak{R})$ is *compact*, that is , every open covering of $Spec(\mathfrak{R})$ has a finite subcovering.

2. $\mathrm{Spec}(\mathfrak{R})$ is T_1-space if and only if every prime ideal of $\mathfrak{R}$ is maximal.

3. $\mathrm{Spec}(\mathfrak{R})$ is irreducible if and only if the nilradical (the set of all nilpotent elements of $\mathfrak{R}$) of $\mathfrak{R}$ is a prime ideal.

4. Let $\phi : \mathfrak{R}_1 \to \mathfrak{R}_2$ be a ring homomorphism, then ϕ induces a continuous map $\phi^* : \mathrm{Spec}(\mathfrak{R}_2) \to \mathrm{Spec}(\mathfrak{R}_1)$, given by $\phi^*(P) = \phi^{-1}(P)$ $(P \in \mathrm{Spec}(\mathfrak{R}_2))$.

5. For a Neotherian ring $\mathfrak{R}$ (every acsending chain of ideals in $\mathfrak{R}$ is stationary), $\mathrm{Spec}(\mathfrak{R})$ is discrete and finite if and only if $\mathfrak{R}$ is an Artinian ring (every desending chain of ideals in $\mathfrak{R}$ is stationary).

A *prime module* is a left $\mathfrak{R}$-module E which satisfies that $Ann(\mathcal{N}) = Ann(E)$

for every nonzero submodule $\mathcal{N}$ of E, where

$$Ann(\mathcal{N}) = \{r \in \mathfrak{R} | \forall n \in \mathcal{N} : rn = 0\}.$$

Then a submodule $\mathcal{P}$ of E is said to be a *prime submodule* if $E/\mathcal{P}$ is a prime module (see [11]).

In 2008, M. Behboodi and M. R. Haddadi started the idea of generalizing the Zariski topology of rings to modules [7, 6]. Unfortunately, not all the modules can have the Zariski topology, since the collection $\{V(\mathcal{N}) | \mathcal{N}$ is a submodule of $E\}$ (where $V(\mathcal{N})$ is the set of all prime submodules of E containing $\mathcal{N}$) is not closed under finite unions. All this made necessary, the introduction of the notion of the *top module*. The module E is called a top module if it has Zariski topology (the collection $\{V(\mathcal{N})\}$ closed under finite unions). The *multiplication modules* make upon example of top modules, where a module E over a commutative ring $\mathfrak{R}$ is a multiplication module if each submodule of E is of the form $\mathcal{I}E$ where $\mathcal{I}$ is an ideal of $\mathfrak{R}$ (see [26, Theorem 3.5]).

In chapter two of this thesis we will reproduce all the results of **M. Behboodi and M. R. Haddadi paper, classical Zariski topology of modules and spectral spaces I** [7]. Mainly, as in chapter one, we will discuss some basic properties of the classical Zariski topology. The following are some discussed facts,

1. For each $\mathfrak{R}$-module E, $\mathrm{Spec}(E)$ is a T_1-space if and only if $dim(E) \leq 0$, where $dim(E)$ is the prime dimension of E [23, 24].

2. $\mathrm{Spec}(E)$ is the cofinite topology if and only if $dim(E) \leq 0$ and for every submodule $\mathcal{N}$ of E either $V(\mathcal{N})$ is the whole space or finite.

3. For each left $\mathfrak{R}$-module E with finite spectrum, $\mathrm{Spec}(E)$ is a spectral space,

where the *spectral space* is a topological space that is homeomorphic to the spectrum of a commutative ring.

In addition to reproducing the results of the papers considered above, we will also use those results to enrich the literature with new examples subject to the topological and the algebraic discussed notions.

As we mentioned above, this topic links three basic branches of pure mathematics, algebra, topology and algebraic geometry. Thus, many algebraic concepts can be reformulated through topological characteristics, as well as, one may appeal to the algebraic notions (of rings or modules) to provide examples and counter examples of topological spaces with specific properties. These are two motivations for studying this topic.

Preliminaries

0.1 Ring and Module theory

In this section we recall some basic definitions and facts that will be used frequently in this thesis. We will use [31, 3] as main references.

0.1.1 Generated ideals

Definition 0.1. Let $\mathfrak{R}$ be a ring and $\mathcal{S}$ be a subset of $\mathfrak{R}$. The *ideal of* $\mathfrak{R}$ *generated by* $\mathcal{S}$ is the smallest ideal of $\mathfrak{R}$ which contains $\mathcal{S}$ and is denoted by $(\mathcal{S})$.

In particular if $\mathcal{S} = \{x\}$, then $(x) = \{rx | r \in \mathfrak{R}\}$. Also, if $\mathcal{S}$ is a finite set, then the ideal $\mathcal{I} = (\mathcal{S})$ of $\mathfrak{R}$ is called a *finitely generated ideal of* $\mathfrak{R}$.
e.g. If $\mathcal{S} = \{x_1, x_2\}$, then $\mathcal{I} = (\mathcal{S}) = \mathfrak{R}x_1 + \mathfrak{R}x_2$.

0.1.2 Prime ideals and Maximal Ideals

Definition 0.2. Let $\mathfrak{R}$ be a ring, then

1. The ideal $\mathcal{P}$ of $\mathfrak{R}$ is a *prime ideal of* $\mathfrak{R}$ if and only if $\mathcal{P} \neq \mathfrak{R}$ and if $\mathcal{A}, \mathcal{B}$ are ideals of $\mathfrak{R}$, $\mathcal{A} \subseteq \mathcal{P}$ or $\mathcal{B} \subseteq \mathcal{P}$ whenever $\mathcal{A}\mathcal{B} \subseteq \mathcal{P}$.

2. If $\mathfrak{R}$ is commutative, then $\mathcal{P}$ is prime if and only if $\mathcal{P} \neq \mathfrak{R}$ and if $a, b \in \mathfrak{R}$, $a \in \mathcal{P}$ or $b \in \mathcal{P}$ whenever $ab \in \mathcal{P}$.

3. The set of all prime ideals of $\mathfrak{R}$ is called *the prime spectrum of* $\mathfrak{R}$ and is denoted by $Spec(\mathfrak{R})$.

4. The ideal $\mathcal{M}$ of $\mathfrak{R}$ is a *maximal ideal of* $\mathfrak{R}$ if and only if there is no ideal $\mathcal{I}$ of $\mathfrak{R}$ that containing $\mathcal{M}$ properly and is contained strictly in $\mathfrak{R}$. The set of all maximal ideals is called *the maximal spectrum of* $\mathfrak{R}$ and denoted by $Max(\mathfrak{R})$.

5. The ideal $\mathcal{L}$ of $\mathfrak{R}$ is a *minimal ideal of* $\mathfrak{R}$ if and only if there is no nonzero ideal $\mathcal{J}$ of $\mathfrak{R}$ strictly contained in $\mathcal{L}$.

6. The ideal P is of $\mathfrak{R}$ is a *minimal prime ideal* if and only if it is a prime ideal, and there is no prime ideal strictly contained inside P.

Proposition 0.3. Let $\mathfrak{R} = \mathbb{Z}_n$. Then

1. the ideals in $\mathbb{Z}_n$ are the sets of the form $\langle d \rangle$ where d divides n.

2. the maximal ideals in $\mathbb{Z}_n$ are the sets of the form $\langle p \rangle$ where p is a prime divides n.

Proposition 0.4. [3, Theorem 1.3] Let $\mathfrak{R}$ be a nonzero commutative ring. Then $\mathfrak{R}$ has at least one maximal ideal.

Proposition 0.5. [3, Crollary 1.4]Let $\mathfrak{R}$ be a commutative ring and let $\mathcal{I}$ be a proper ideal of $\mathfrak{R}$, then there exists a maximal ideal $\mathcal{M}$ of $\mathfrak{R}$ containing $\mathcal{I}$.

Lemma 0.6. [31, Lemma 3.23, Lemma 3.3] Let $\mathfrak{R}$ be a commutative ring and $\mathcal{I}$ an ideal of $\mathfrak{R}$. Then

1. $\mathcal{I}$ is prime if and only if $\mathfrak{R}/\mathcal{I}$ is an integral domain.

2. $\mathcal{I}$ is maximal if and only if $\mathfrak{R}/\mathcal{I}$ is a field.

0.1.3 Radicals

Definition 0.7. Let $\mathfrak{R}$ be a ring and $a \in R$.

1. If there is $n \geq 1$ such that $a^n = 0$, then a called a nilpotent element. Note that 0 is a nilpotent element in any ring. The rings with no nonzero nilpotent element is called a *reduced ring*. e.g. Integral domains are reduced ring.

2. The set of all nilpotents of $\mathfrak{R}$ is an ideal of $\mathfrak{R}$ called *the nilradical of* $\mathfrak{R}$ and denoted by $Nil(\mathfrak{R})$. Equivalently,

$$Nil(\mathfrak{R}) = \bigcap_{\mathcal{P} \in Spec(\mathfrak{R})} \mathcal{P}.$$

3. If $a^2 = a$, then a is called an *idempotent element* of $\mathfrak{R}$.

Definition 0.8. Let $\mathfrak{R}$ be a ring and $\mathcal{I}$ an ideal of $\mathfrak{R}$. The *radical of* $\mathcal{I}$, $\sqrt{\mathcal{I}}$ is

$$\sqrt{\mathcal{I}} = \{x \in \mathfrak{R} | x^n \in \mathcal{I} \textit{ for some } n \geq 1\} = \bigcap \{\mathcal{P} \in Spec(\mathfrak{R}) | \mathcal{I} \subseteq \mathcal{P}\}.$$

Notice that $Nil(\mathfrak{R}) = \sqrt{(0)}$ where (0) is the zero ideal of $\mathfrak{R}$.

0.1.4 Dimension of a ring

Definition 0.9. [31, Definition 14.17] Let $\mathfrak{R}$ be a commutative ring, and $\mathcal{P}_0, ..., \mathcal{P}_n$ are prime ideals of $\mathfrak{R}$,

1. The chain $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_n$ is called a chain of prime ideals of $\mathfrak{R}$ of length n.

2. If $\mathcal{P} \in Spec(\mathfrak{R})$. Then the height of $\mathcal{P}$, denoted by $ht(\mathcal{P})$ is defined to be the supremum of lengths of chains $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \cdots \subsetneq \mathcal{P}_n = \mathcal{P}$ if this supremum exists, and ∞ otherwise.

3. The dimension of $\mathfrak{R}$, denoted by $dim(\mathfrak{R})$,

$$dim(\mathfrak{R}) = \sup\{ht(\mathcal{P}) | \mathcal{P} \in Spec(\mathfrak{R})\}.$$

e.g. Let F be a field, $dim(F) = 0$ since the only prime ideal is (0). For the ring of integers $\mathbb{Z}$, $dim(\mathbb{Z}) = 1$, see [31, Examples 14.19 (iii)].

0.1.5 Noetherian and Artinian rings

Definition 0.10. Let $\mathfrak{R}$ be a commutative ring.

1. The ring $\mathfrak{R}$ is a Noetherian ring if and only if every ascending chain of ideals of $\mathfrak{R}$, $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots$, is stationary. Equivalently, every ideal of $\mathfrak{R}$ is finitely generated.

2. The ring $\mathfrak{R}$ is an Artinian ring if and only if every descending chain of ideals of $\mathfrak{R}$, $\mathcal{I}_1 \supseteq \mathcal{I}_2 \supseteq \cdots$, is stationary.

Proposition 0.11. Let $\mathfrak{R}$ be a ring.

1. If $\mathfrak{R}$ is an Artinian ring, then every prime ideal is maximal.

2. If $\mathfrak{R}$ is an Artinian ring, then the number of the maximal ideals of $\mathfrak{R}$ is finite.

3. $\mathfrak{R}$ is Artinian if and only if $\mathfrak{R}$ is Noetherian and $dim(\mathfrak{R}) = 0$.

0.1.6 Localization

Let $\mathfrak{R}$ be a commutative ring.

Definition 0.12. [31] Let $\mathfrak{R}$ be a commutative ring and $\mathcal{S}$ be a subset of $\mathfrak{R}$. The subset $\mathcal{S}$ is called a *multiplicative closed* subset of $\mathfrak{R}$ if:

1. $1_{\mathfrak{R}} \in \mathcal{S}$, and

2. $a, b \in \mathcal{S}$ implies that $ab \in \mathcal{S}$.

e.g. (1) Let $\mathfrak{R}$ be an integral domain. Then $\mathcal{S} = \mathfrak{R} \setminus \{0\}$ is a multiplicative closed set of $\mathfrak{R}$.
(2) Let $\mathfrak{R}$ be a commutative ring and $\mathcal{P}$ be a prime ideal of $\mathfrak{R}$. Then $\mathcal{S} = \mathfrak{R} \setminus \mathcal{P}$ is a multiplicative closed subset of $\mathfrak{R}$.

Definition 0.13. [31] Let $\mathfrak{R}$ be a commutative ring and $\mathcal{S} \subseteq \mathfrak{R}$ be a multiplicative closed set of $\mathfrak{R}$. Then the relation $\sim$ on $\mathfrak{R} \times \mathcal{S}$ which defined by

$$(a, s) \sim (a', s') \text{ if and only if } \exists\, u \in \mathcal{S} \text{ such that } u(as' - a's) = 0$$

is an eqivalent relation.

The equivalece classes (a, s) are denoted by $\frac{a}{s}$ where $a \in \mathfrak{R}$ and $s \in \mathcal{S}$. The set of all equivalece classes

$$\mathcal{S}^{-1}\mathfrak{R} = \{\frac{a}{s} \mid a \in \mathfrak{R}, s \in \mathcal{S}\}$$

is called the *localization of* $\mathfrak{R}$ at the multiplicative closed set $\mathcal{S}$. The set $\mathcal{S}^{-1}\mathfrak{R}$

is form a ring with

$$+ : \frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1 s_2 + a_2 s_1}{s_1 s_2}$$
$$\cdot : \frac{a_1}{s_1} \cdot \frac{a_2}{s_2} = \frac{a_1 a_2}{s_1 s_2}$$

Note that, $1_{\mathcal{S}^{-1}\mathfrak{R}} = \frac{1}{1}$ and $0_{\mathcal{S}^{-1}\mathfrak{R}} = \frac{0}{1}$.

Particular Case: If $\mathfrak{R}$ is an integral domain and $\mathcal{S} = \mathfrak{R} \setminus \{0\}$, then $\mathcal{S}^{-1}\mathfrak{R}$ is called the *qoutient field* or *field of fraction* of $\mathfrak{R}$. Note that, every nonzero element $\frac{a}{s}$ is a unit with inverse $\frac{s}{a}$.

Let $\mathfrak{R}$ be a ring and $\mathcal{S} = \mathfrak{R} \setminus \mathcal{P}$, where $\mathcal{P}$ is a prime ideal of $\mathfrak{R}$. Then

$$\mathcal{S}^{-1}\mathfrak{R} = \mathfrak{R}_{\mathcal{P}} = \{\frac{a}{s} \mid a \in \mathfrak{R}, s \notin \mathcal{P}\}$$

Proposition 0.14. [31] Let $\mathfrak{R}$ be a ring and $\mathcal{S}$ be a multiplicative closed set of $\mathfrak{R}$. Then

1. Any ideal of $\mathcal{S}^{-1}\mathfrak{R}$ is of the form $\mathcal{S}^{-1}\mathcal{J}$ for some ideal $\mathcal{J}$ of $\mathfrak{R}$.

2. Prime ideals of $\mathcal{S}^{-1}\mathfrak{R}$ is of the form $\mathcal{S}^{-1}\mathcal{P}$ with $\mathcal{P} \cap \mathcal{S} = \emptyset$ and $\mathcal{P}$ is a prime ideal of $\mathfrak{R}$.

Proposition 0.15. [31] Let $\mathfrak{R}$ be a ring and $\mathcal{S} = \mathfrak{R} \setminus \mathcal{P}$, then $\mathfrak{R}_{\mathcal{P}}$ is a local ring with maximal $\mathcal{S}^{-1}\mathcal{P} = \mathcal{P}\mathfrak{R}_{\mathcal{P}} = Nil(\mathfrak{R}_{\mathcal{P}})$.

0.1.7 Rings homomorphism

Definition 0.16. Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be two rings and $\theta : \mathfrak{R}_1 \to \mathfrak{R}_2$ be a mapping from $\mathfrak{R}_1$ into $\mathfrak{R}_2$ such that

(1) $\theta(x + y) = \theta(x) + \theta(y)$ for all $x, y \in \mathfrak{R}_1$,

(2) $\theta(xy) = \theta(x)\theta(y)$ for all $x, y \in \mathfrak{R}_1$,

(3) $\theta(1_{\mathfrak{R}_1}) = 1_{\mathfrak{R}_2}$.

Then θ is called a *ring homomorphism*. If θ is a bijection map, then it is called an *isomorphism* and we say that $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are isomorphic, in this case we write $\mathfrak{R}_1 \cong \mathfrak{R}_2$.

Proposition 0.17. Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be two rings and $\theta : \mathfrak{R}_1 \to \mathfrak{R}_2$ be a ring homomorphism.

(1) The kernel of θ defined by $Ker(\theta) = \{r \in \mathfrak{R}_1 | \theta(r) = 0\}$ is an ideal of $\mathfrak{R}_1$.

(2) The image of θ, denoted $Im(\theta)$, is a subring of $\mathfrak{R}_2$.

(3) The homomorphism θ is injective if and only if $Ker(\theta) = 0_{\mathfrak{R}_1}$.

(4) The homomorphism θ is surjective if and only if $Im(\theta) = \mathfrak{R}_2$.

(5) The image of θ is isomorphic to the quotient ring $\mathfrak{R}_1/Ker(\theta)$. If θ is surjective then $\mathfrak{R}_2$ is isomorphic to $\mathfrak{R}_1/ker(\theta)$.

(6) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are commutative and $\mathcal{I}$ is an ideal of $\mathfrak{R}_2$, then $\theta^{-1}(\mathcal{I})$ is an ideal of $\mathfrak{R}_1$.

(7) If $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are commutative and $\mathcal{P}$ is a prime ideal of $\mathfrak{R}_2$, then $\theta^{-1}(\mathcal{P})$ is a prime ideal of $\mathfrak{R}_1$.

0.1.8 Modules

Throughout, all rings are with identity element and all modules are unitary left modules.

Definition 0.18. Let E be an $\mathfrak{R}$-module.

1. The *annihilator* of a nonempty subset $\mathcal{S}$ of E is an ideal of $\mathfrak{R}$ denoted by $Ann_{\mathfrak{R}}(\mathcal{S})$ is
$$Ann_{\mathfrak{R}}(\mathcal{S}) = \{r \in \mathfrak{R} | r\mathcal{S} = 0_E\}.$$

2. For any submodule $\mathcal{N}$ of E we use $(\mathcal{N} : E)$ to denote $Ann_{\mathfrak{R}}(E/\mathcal{N})$,
$$Ann_{\mathfrak{R}}(E/\mathcal{N}) = \{r \in \mathfrak{R} | r(E/\mathcal{N}) = 0/\mathcal{N}\},$$

   Now, $rE + N = 0 + N$ implies $rE \subseteq N$ thus
$$(\mathcal{N} : E) = \{r \in \mathfrak{R} : rE \subseteq \mathcal{N}\}.$$

   $(\mathcal{N} : E)$ is called the *residual of* $\mathcal{N}$ by E.

0.1.9 Finitely generated modules

Definition 0.19. Let $\mathfrak{R}$ be a ring and E an $\mathfrak{R}$-module.

1. If $x \in E$, then the set $\mathfrak{R}x = \{rx \mid r \in \mathfrak{R}\}$ is a submodule of E and it is denoted by (x).

2. If $\{x_i\}_{i \in I}$ are elements of E and $E = \sum_{i \in I} \mathfrak{R}x_i$, then $\{x_i\}_{i \in I}$ are called the *set of generators of* E.

3. If $\{x_i\}_{i \in I}$ is a finite set of E, then E is called a *finitely generated module*.

0.1.10 Noetherian and Artinian modules

Definition 0.20. Let $\mathfrak{R}$ be a ring and E an $\mathfrak{R}$-module.

1. The module E is a Noetherian module if and only if every ascending chain of submodules of E, $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \cdots$, is stationary. Equivalently, every submodule of E is finitely generated.

2. The module E is an Artinian module if and only if every descending chain of submodules of E, $\mathcal{N}_1 \supseteq \mathcal{N}_2 \supseteq \cdots$, is stationary.

0.2 General Topology

In this section we recall some basic topological definitions. The reader can refer to [29, 17].

Definition 0.21. Let T be a nonempty set. A topology on T is a nonempty family $\mathcal{T}$ of subsets of T which satisfies the following conditions:

- $\emptyset, T \in \mathcal{T}$.
- If $\{A_i\}_{i\in I} \in \mathcal{T}$ then $\bigcap_{i\in I} A_i \in \mathcal{T}$.
- If $A, B \in \mathcal{T}$, then $A \cup B \in \mathcal{T}$.

The subsets of T belonging to the family $\mathcal{T}$ are called closed sets. The set T together with a topology $\mathcal{T}$ is called topological space and is denoted by $(T, \mathcal{T})$ or simply by T. The complement of the elements of $\mathcal{T}$ is called open sets.

Definition 0.22. Let $\mathcal{T}$ be a topology on the nonempty set T. A basis of $\mathcal{T}$ is a family of open sets $\mathcal{O} \subset \mathcal{T}$ such that every open set of $\mathcal{T}$ is a union of elements of $\mathcal{O}$.

Definition 0.23. Let $\mathcal{T}$ be a topology on the nonempty set T. A subbasis of $\mathcal{T}$ is a family of open sets $S \subset \mathcal{T}$ such that the family $\mathcal{O}_S := \{U_1 \cap U_2 \cap \cdots \cap U_n | U_1, U_2, \cdots, U_n \in S\}$ is a basis of $\mathcal{T}$.

Definition 0.24. Let S be a subset of a topological space T. The closure of S, $\overline{S}$, is the intersection of all the closed subsets of T which contain S (is the smallest closed set containing S).

Proposition 0.25. Let $\mathcal{S}$ be a subset of a topological space T. Then:

1. $\mathcal{S} \subseteq \overline{\mathcal{S}}$.

2. $\mathcal{S}$ is closed if and only if $\mathcal{S} = \overline{\mathcal{S}}$.

3. If $\mathcal{L}$ is a subset of T such that $\mathcal{S} \subseteq \mathcal{L}$, then $\overline{\mathcal{S}} \subseteq \overline{\mathcal{L}}$.

4. If $\mathcal{S} \subseteq \mathcal{L}$ and $\mathcal{L}$ is closed, then $\overline{\mathcal{S}} \subseteq \mathcal{L}$.

Let $(T, \mathcal{T})$ be a topological space and let $\mathcal{S}$ be a nonempty subset of T. If we consider the family $\mathcal{T}_{\mathcal{S}}$ of subsets of $\mathcal{S}$ defined as follows $\mathcal{T}_{\mathcal{S}} := \{\mathcal{S} \cap \mathcal{A} | \mathcal{A} \in \mathcal{T}\}$, $\mathcal{T}_{\mathcal{S}}$ is a topology on $\mathcal{S}$. $\mathcal{T}_{\mathcal{S}}$ the topology induced on $\mathcal{S}$ by T (or subspace topology) and $(\mathcal{S}, \mathcal{T}_{\mathcal{S}})$ will be called topological subspace.

0.2.1 Separation Axioms

A common way to classify different classes of topological spaces is by using the *separation axioms*.

Definition 0.26. Let T be a topological space. We say that T is a

- T_0-space if for any $a \neq b \in T$ there exist U, V open subsets of T such that $a \in U,\ b \notin U$ or $b \in V,\ a \notin V$.

- T_1-space if for any $a \neq b \in T$ there exist U, V open subsets of T such that $a \in U,\ b \notin U$ and $b \in V,\ a \notin V$.

- Hausdorff or T_2-space if for any $a \neq b \in T$ there exist U, V disjoint open subsets of T such that $a \in U$ and $b \in V$.

Note that a Hausdorff space is also a T_1 space and that a T_1 space is also a T_0 space.

Lemma 0.27. Let T be a topological space. Then, T is a T_0-space if and only if for any $a, b \in T$, $\overline{\{a\}} = \overline{\{b\}}$ implies $a = b$.

Lemma 0.28. Let T be a topological space. Then, T is a T_1-space if and only if every singleton of T is closed.

Definition 0.29. Let T be a set. A collection $\{A_i\}_{i \in I}$ of subsets of T is called a cover of T if $T = \bigcup_{i \in I} A_i$. If $\{A_i\}_{i \in I}$ are open sets, then it is called an open cover of T.

Definition 0.30. A topological space T is called compact if every open cover of T contains a finite subcover, i.e.,

$$T = \bigcup_{i \in I} A_i \implies T = \bigcup_{j=1}^{n} A_j$$

A subset S of the space T is called compact if it is a compact space with respect to the subspace topology.

Definition 0.31. [3] A nonempty topological space T such that every pair of nonempty open sets in T intersect is called *irreducible space*.

Definition 0.32. The maximal irreducible subspaces of T are called the *irreducible components of* T.

The next proposition presents the properties of irreducible spaces.

Proposition 0.33. [9, Page 95] Let T be a topological space.

(i) If $\mathcal{A}$ is an irreducible subspace of T, then $\overline{\mathcal{A}}$ is irreducible.

(ii) Every irreducible subspace of T is contained in a maximal irreducible subspace.

(iii) The irreducible components of T are closed and cover T.

Proposition 0.34. [9] Let $f : T \to T'$ be a continuous map of topological spaces. If $E \subset T$ is an irreducible subset, then $f(E) \subset T'$ is irreducible.

Definition 0.35. Let T be a topological space. An element a in the closed subset $\mathcal{A}$ is called a *generic point* of $\mathcal{A}$ if $\mathcal{A} = \overline{\{a\}}$.

Definition 0.36. [3] A topological space T is called Noetherian space if it's closed subsets satisfy the descending chain condition; i.e. for every sequence $Y_1 \supseteq Y_2 \supseteq Y_3 \cdots$ of closed subsets Y_i of T, $Y_n = Y_{n+1} = \cdots$ for some n. Equivalently, the open subsets satisfy the ascending chain condition.

Proposition 0.37. [3] Let T be a topological space.

(1) If T is a Noetherian space, then it is compact.

(2) If T is a Noetherian space, then every subspace of T is Noetherian.

(3) T is a Noetherian space if and only if every open subspace of T is compact.

(4) T is a Noetherian space if and only if every subspace of T is compact.

Definition 0.38. If T is a topological space and $p \in T$, a neighborhood of p is a subset V of T that includes an open set U containing p, $p \in U \subseteq V$. The neighborhood V need not be an open set itself. If V is open it is called an open neighborhood.

Definition 0.39. Let T, S be topological spaces. A function $\mathfrak{f} : T \to S$ is called continuous at the point $x \in T$ if for each neighborhood N of $\mathfrak{f}(x) \in S$ there exists a neighborhood M of x such that $\mathfrak{f}(M) \subset N$. The function $\mathfrak{f}$ is called continuous on T if it is continuous at every point of T.

Proposition 0.40. Let $\mathfrak{f} : T \to S$ be a function between topological spaces. Then, the following assertions are equivalent:

(1) $\mathfrak{f}$ is continuous.

(2) $\mathfrak{f}^{-1}(\mathcal{A})$ is open in T for each open subset $\mathcal{A}$ of S.

(3) $\mathfrak{f}^{-1}(\mathcal{B})$ is closed in T for each closed subset $\mathcal{B}$ of S.

Definition 0.41. Let T, S be topological spaces. A function $\mathfrak{f} : T \to S$ is called an homeomorphism if $\mathfrak{f}$ is continuous, bijective and $\mathfrak{f}^{-1} : S \to T$ is continuous. Two topological spaces are called homeomorphic if there exists a homeomorphism between them.

Definition 0.42. The topological space T is called a *disconnected space* if it can be decomposed as a disjoint union of two nonempty closed subsets.

CHAPTER 1

Zariski topology of rings

Throughout this chapter, all rings are commutative with unity.

1.1 Definition and Examples

Throughout this section, $\mathfrak{R}$ denotes a commutative ring with unity, $\mathcal{S}$ denotes a subset of $\mathfrak{R}$, $\mathcal{I}$ denotes an ideal of $\mathfrak{R}$, $\sqrt{\mathcal{I}}$ is the radical of $\mathcal{I}$ and $Spec(\mathfrak{R})$ is the set of all prime ideals of $\mathfrak{R}$. We start by the following definition.

Definition 1.1. Let $\mathcal{S}$ be a subset of $\mathfrak{R}$. The variety of $\mathcal{S}$, $V(\mathcal{S})$, defined by

$$V(\mathcal{S}) = \{\mathcal{P} \in Spec(\mathfrak{R}) | \mathcal{S} \subseteq \mathcal{P}\}.$$

Example 1.2. Let $\mathfrak{R} = \mathbb{Z}$. Then $\mathrm{Spec}(\mathbb{Z}) = \{(0), (p) :\ p$ is prime integer$\}$ by [3, page 4 Examples]. Now, if $\mathcal{S} = (6)$. Then $V(\mathcal{S}) = \{(2), (3)\}$.

The next Lemma proves that any variety in $\mathfrak{R}$ is of the form $V(\sqrt{\mathcal{I}})$ for some ideal $\mathcal{I}$ of $\mathfrak{R}$.

Lemma 1.3. Let $\mathfrak{R}$ be a ring and $\mathcal{S}$ a subset of $\mathfrak{R}$. If $\mathcal{I}$ is the ideal in $\mathfrak{R}$ generated by $\mathcal{S}$, then

$$V(\mathcal{S}) = V(\mathcal{I}) = V(\sqrt{\mathcal{I}}).$$

Proof. For the first equality, let $\mathcal{P} \in V(\mathcal{S})$, then $\mathcal{S} \subseteq \mathcal{P}$. Now, $\mathcal{I} \subseteq \mathcal{P}$ since $\mathcal{I}$ is the smallest ideal that contains $\mathcal{S}$, hence $\mathcal{P} \in V(\mathcal{I})$. For the second inclusion, let $\mathcal{P} \in V(\mathcal{I})$, then $\mathcal{I} \subseteq \mathcal{P}$. But $\mathcal{S} \subseteq \mathcal{I}$. Thus, $\mathcal{S} \subseteq \mathcal{P}$, i.e. $\mathcal{P} \in V(\mathcal{S})$.

Now, for the second equality, if $\mathcal{P} \in V(\mathcal{I})$ and $x \in \sqrt{\mathcal{I}}$ then there is $k \in \mathbb{N}$ such that $x^k \in \mathcal{I}$. Now, $\mathcal{P} \in V(\mathcal{I})$ implies $\mathcal{I} \subseteq \mathcal{P}$. Thus $x^k \in \mathcal{P}$ and this implies that $x \in \mathcal{P}$ since $\mathcal{P}$ is prime. Therefore $\sqrt{\mathcal{I}} \subseteq \mathcal{P}$. The second inclusion is true since $\mathcal{I} \subseteq \sqrt{\mathcal{I}}$. □

Next we discuss some properties of the varieties in $\mathfrak{R}$.

Lemma 1.4. Suppose that $\mathcal{I}, \mathcal{J} \subset \mathfrak{R}$ are ideals in $\mathfrak{R}$. Then

(i) If $\mathcal{I} \subset \mathcal{J}$, then $V(\mathcal{J}) \subset V(\mathcal{I})$.

(ii) $V(\mathcal{I}) \subset V(\mathcal{J})$ if and only if $\sqrt{\mathcal{J}} \subset \sqrt{\mathcal{I}}$.

(iii) $V(\mathcal{I}) = V(\mathcal{J})$ if and only if $\sqrt{\mathcal{J}} = \sqrt{\mathcal{I}}$.

(iv) $V(\mathcal{I} \cap \mathcal{J}) = V(\mathcal{I}\mathcal{J}) = V(\mathcal{I}) \cup V(\mathcal{J})$.

Proof. (i) Suppose that $\mathcal{I} \subset \mathcal{J}$ are ideals of $\mathfrak{R}$, if $\mathcal{P}$ is prime ideal in $V(\mathcal{J})$, then $\mathcal{J} \subseteq \mathcal{P}$. Therefore, $\mathcal{I} \subseteq \mathcal{P}$ and $V(\mathcal{J}) \subset V(\mathcal{I})$.

(ii) First, assume that $V(\mathcal{I}) \subset V(\mathcal{J})$ which implies that for every prime ideal $\mathcal{P}$ of $\mathfrak{R}$ such that $\mathcal{I} \subset \mathcal{P}$, then $\mathcal{J} \subset \mathcal{P}$. Therefore, if $x \in \sqrt{\mathcal{J}}$, then since $\sqrt{\mathcal{J}}$ is the intersection of all prime ideals containing $\mathcal{J}$, and if $\mathcal{P}$ is a prime ideal such that $\mathcal{I} \subset \mathcal{P}$ implies that $\mathcal{J} \subset \mathcal{P}$, then $x \in \mathcal{P}$, hence $x \in \sqrt{\mathcal{I}}$ and $\sqrt{\mathcal{J}} \subset \sqrt{\mathcal{I}}$. Now for the second direction, suppose $\sqrt{\mathcal{J}} \subset \sqrt{\mathcal{I}}$, then by (i) and Lemma 1.3, $V(\mathcal{I}) \subset V(\mathcal{J})$.

(iii) Suppose $V(\mathcal{I}) = V(\mathcal{J})$, then $V(\mathcal{I}) \subset V(\mathcal{J})$ implies by (ii) $\sqrt{\mathcal{J}} \subset \sqrt{\mathcal{I}}$. Also, $V(\mathcal{J}) \subset V(\mathcal{I})$ implies $\sqrt{\mathcal{I}} \subset \sqrt{\mathcal{J}}$. Hence, $\sqrt{\mathcal{J}} = \sqrt{\mathcal{I}}$. Conversely, if $\sqrt{\mathcal{J}} = \sqrt{\mathcal{I}}$, then by Lemma 1.3, $V(\mathcal{J}) = V(\sqrt{\mathcal{J}}) = V(\sqrt{\mathcal{I}}) = V(\mathcal{I})$.

(iv) For $\mathcal{I}, \mathcal{J}$ ideals of $\mathfrak{R}$, $\mathcal{I}\mathcal{J} \subset \mathcal{I} \cap \mathcal{J} \subset \mathcal{I}$ and $\mathcal{I}\mathcal{J} \subset \mathcal{I} \cap \mathcal{J} \subset \mathcal{J}$. By (i), $V(\mathcal{I}) \subset V(\mathcal{I} \cap \mathcal{J}) \subset V(\mathcal{I}\mathcal{J})$ and $V(\mathcal{J}) \subset V(\mathcal{I} \cap \mathcal{J}) \subset V(\mathcal{I}\mathcal{J})$. Hence $V(\mathcal{I}) \cup V(\mathcal{J}) \subset V(\mathcal{I} \cap \mathcal{J}) \subset V(\mathcal{I}\mathcal{J})$. Now, let $\mathcal{P} \in V(\mathcal{I}\mathcal{J})$ so $\mathcal{I}\mathcal{J} \subset \mathcal{P}$. Then $\mathcal{I} \subset \mathcal{P}$ or $\mathcal{J} \subset \mathcal{P}$ since $\mathcal{P}$ is prime ideal. Therefore $\mathcal{P} \in V(\mathcal{I})$ or $\mathcal{P} \in V(\mathcal{J})$. Hence $\mathcal{P} \in V(\mathcal{I}) \cup V(\mathcal{J})$. i.e. $V(\mathcal{I}) \cup V(\mathcal{J}) \subset V(\mathcal{I} \cap \mathcal{J}) \subset V(\mathcal{I}\mathcal{J}) \subset V(\mathcal{I}) \cup V(\mathcal{J})$ □

Next, we will see how the varieties of a ring $\mathfrak{R}$ can be used to define a topology on $Spec(\mathfrak{R})$.

Proposition 1.5. Let $\mathfrak{R}$ be a ring. Then $Spec(\mathfrak{R})$ has a topology whose closed sets are the collection $\mathcal{V}(\mathfrak{R}) = \{V(\mathcal{S}) | \mathcal{S} \text{ subset of } \mathfrak{R}\}$. This topology is called the *Zariski topology* of $\mathfrak{R}$ and denoted by τ.

Proof. To prove that $(Spec(\mathfrak{R}), \tau)$ forms a topological space, we show that the sets $V(\mathcal{S})$ satisfy the axioms for closed sets in the space $Spec(\mathfrak{R})$.

1. The empty set and the space $Spec(\mathfrak{R})$ are closed sets, because for every prime ideal $\mathcal{P}$, $0 \in \mathcal{P}$ and $1_{\mathfrak{R}} \notin \mathcal{P}$ since $\mathcal{P}$ is proper. Hence, $V(0) = Spec(\mathfrak{R})$ and $V(1_{\mathfrak{R}}) = \emptyset$.

2. The intersection of any family of closed sets in $Spec(\mathfrak{R})$ is closed, if $(\mathcal{S}_i)_{i\in I}$ is a family of subsets of $\mathfrak{R}$, then $\mathcal{S}_i \subset \bigcup_{i\in I} \mathcal{S}_i$ for all $i \in I$. By Lemma 1.4 (i), $V(\bigcup_{i\in I} \mathcal{S}_i) \subset V(\mathcal{S}_i)$ for all $i \in I$. Thus $V(\bigcup_{i\in I} \mathcal{S}_i) \subset \bigcap_{i\in I} V(\mathcal{S}_i)$. Now, if $\mathcal{P} \in V(\mathcal{S}_i)$ for all $i \in I$, $\mathcal{S}_i \subset \mathcal{P}$ for all $i \in I$. Hence, $\bigcup_{i\in I} \mathcal{S}_i \subset \mathcal{P}$ and $\mathcal{P} \in V(\bigcup_{i\in I} \mathcal{S}_i)$. Therefore, $V(\bigcup_{i\in I} \mathcal{S}_i) = \bigcap_{i\in I} V(\mathcal{S}_i)$.

3. The finite union of closed sets is closed. By Lemma 1.4 (iv) □

Next, we provide the Zariski topology of some special rings.

Examples 1.6. 1. **For any field** $\mathbb{F}$ the only prime ideal is (0). Hence, the closed sets in $\mathrm{Spec}\,(\mathbb{F})$ are $\emptyset$ and $\mathrm{Spec}\,(\mathbb{F})$. i.e. the Zariski topology on $\mathrm{Spec}\,(\mathbb{F})$ is the trivial topology.

2. **For the ring of integers** $\mathbb{Z}$: $\mathrm{Spec}\,(\mathbb{Z}) = \{(0),(2),(3),(5),(7),(11),\cdots\}$. Now, we show that any finite subset of $\mathrm{Spec}\,\mathbb{Z}$ under Zariski topology is closed. Suppose that $\mathcal{S} = \{(p_1),(p_2),\cdots,(p_n)\}$ is a finite collection of prime ideals in $\mathbb{Z}$ where $p_i \neq 0$ for all $1 \leq i \leq n$. Now, let $m = p_1.p_2.\cdots.p_n$ and $\mathcal{I} = (m)$, then $V(\mathcal{I}) = \{(p_1),(p_2),\cdots,(p_n)\}$. On the other hand, since $\mathbb{Z}$ is a PID , then for any proper ideal $\mathcal{I} \subset \mathbb{Z}$, there exist a unique $m \geq 2 \in \mathbb{N}$ such that $\mathcal{I} = (m)$. For m, by the fundamental theorem of arithmetic, there exists a unique prime factorization $m = {p_1}^{i_1}{p_2}^{i_2}\cdots{p_n}^{i_n}, i_k \geq 1$. Then for all $1 \leq k \leq n$, the ideal $(p_k) \in \mathrm{Spec}\,(\mathbb{Z})$ and $V(\mathcal{I}) = V((m)) = \{(p_1),(p_2),\cdots,(p_n)\}$. That is to say the nontrivial closed sets in $\mathrm{Spec}\,(\mathbb{Z})$ is a finite subset of $\mathrm{Spec}\,(\mathbb{Z})$. Notice that $\{(0)\}$ is not a closed set.

The following proposition describes the basic open sets of $Spec(\mathfrak{R})$.

Proposition 1.7. Let $W(x)$ denotes $Spec(\mathfrak{R}) - V(x)$, where $x \in \mathfrak{R}$. i.e.

$$W(x) = \{\mathcal{P} \in Spec(\mathfrak{R}) \mid x \notin \mathcal{P}\}.$$

Then the collection $\{W(x)|x \in \mathfrak{R}\}$ forms a basis of open sets for the Zariski topology of $\mathfrak{R}$ and the sets $W(x)$ are called the basic open sets of $Spec(\mathfrak{R})$.

Proof. Let O be any open set in $Spec(\mathfrak{R})$. There is some $\mathcal{S} \subset \mathfrak{R}$ such that $O = Spec(\mathfrak{R}) - V(\mathcal{S}) = Spec(\mathfrak{R}) - \bigcap_{x\in\mathcal{S}} V(x) = \bigcup_{x\in\mathcal{S}} [Spec(\mathfrak{R}) - V(x)] = \bigcup_{x\in\mathcal{S}} W(x)$. Hence, every open set can be written as a union of the sets $W(x)$.

By Definition 0.22, the collection $\{W(x)|x \in \mathfrak{R}\}$ forms a basis of open sets for the Zariski topology of $\mathfrak{R}$. □

The following Proposition discuss some properties of the open basic sets of $Spec(\mathfrak{R})$.

Proposition 1.8. Let $\mathfrak{R}$ be a ring and $x,\ y \in \mathfrak{R}$. Then

(i) $W(x) \cap W(y) = W(xy)$;

(ii) $W(x) = \emptyset$ if and only if x is a nilpotent.

(iii) $W(x) = W(y)$ if and only if $\sqrt{(x)} = \sqrt{(y)}$.

(iv) $W(x) = Spec(\mathfrak{R})$ if and only if x is a unit.

Proof. (i) Suppose $\mathcal{P}$ is a prime ideal such that $\mathcal{P} \in W(x) \cap W(y)$,then $\mathcal{P} \in W(x)$ and $\mathcal{P} \in W(y)$. Now by definition of open basic sets, $x \notin \mathcal{P}$ and $y \notin \mathcal{P}$. Therefore $xy \notin \mathcal{P}$ thus $\mathcal{P} \in W(xy)$. Now, let $\mathcal{P} \in W(xy)$, then $xy \notin \mathcal{P}$. Since $(xy) \subset (x)$ and $(xy) \subset (y)$, we have $(x) \notin \mathcal{P}$ and $(y) \notin \mathcal{P}$. Therefore, $\mathcal{P} \in W(x)$ and $\mathcal{P} \in W(y)$ hence $\mathcal{P} \in W(x) \cap W(y)$.

(ii) Let $W(x) = \emptyset$. Equivalently, $V(x) = Spec(\mathfrak{R})$ which means that $x \in \mathcal{P}, \forall \mathcal{P}$. Hence, $x \in \bigcap_{\mathcal{P} \in Spec(\mathfrak{R})} \mathcal{P} = Nil(\mathfrak{R})$. Therefore, x is a nilpotent element.

(iii) Suppose $W(x) = W(y)$ which equivalent to $V(x) = V(y)$, so by Lemma 1.3, $V(\sqrt{(x)}) = V(\sqrt{(y)})$. Therefore by Lemma 1.4.(iii), $\sqrt{(x)} = \sqrt{(y)}$.

(iv) Let $W(x) = Spec(\mathfrak{R})$. Thus, $W(x) = W(1)$ by (iii) $\sqrt{(x)} = \sqrt{(1_{\mathfrak{R}})} = 1_{\mathfrak{R}}$. Therefore, $1_{\mathfrak{R}} \in (x)$, so x is a unit. □

1.2 Properties of the Zariski topology of a ring

In this section we will discuss some topological properties of $Spec(\mathfrak{R})$ under Zariski topology. First, we discuss the compactness of $Spec(\mathfrak{R})$. The following proposition proves that $Spec(\mathfrak{R})$ with the Zariski topology is always a compact space for any ring $\mathfrak{R}$.

Proposition 1.9. Let $\mathfrak{R}$ be a ring. Then $Spec(\mathfrak{R})$ is a compact space under the Zariski topology.

Proof. Let the open sets $W(\mathcal{I}_i), i \in I$ be a covering of $Spec(\mathfrak{R})$ where each $\mathcal{I}_i$ is an ideal of $\mathfrak{R}$. Thus $Spec(\mathfrak{R}) = \bigcup_{i \in I} W(\mathcal{I}_i) = \bigcup_{i \in I} [Spec(\mathfrak{R}) - V(\mathcal{I}_i)] = Spec(\mathfrak{R}) - \bigcap_{i \in I} V(\mathcal{I}_i) = Spec(\mathfrak{R}) - V(\sum_{i \in I} \mathcal{I}_i)$, since $\sum_{i \in I} \mathcal{I}_i$ is the smallest ideal containing all the ideals $\mathcal{I}_i$. Therefore, $V(\sum_{i \in I} \mathcal{I}_i) = \emptyset = V(1_{\mathfrak{R}})$. Now, there exists a finite subset J of I such that for every $j \in J, 1_{\mathfrak{R}} = \sum_{j \in J} a_j\, x_j$ where $a_j \in \mathfrak{R}$ and $x_j \in \mathcal{I}_j$. Hence, $\sum_{j \in J} \mathcal{I}_j = \mathfrak{R}$ thus $V(\sum_{j \in J} \mathcal{I}_j) = \emptyset$. Thus, $Spec(\mathfrak{R}) = \bigcup_{j \in J} W(\mathcal{I}_j)$. Hence, $\{W(\mathcal{I}_i)\}_{i \in I}$ restricts to a finite subcover. It follows that $Spec(\mathfrak{R})$ is compact. □

The following proposition is a characterization of the compact open subsets of $Spec(\mathfrak{R})$.

Proposition 1.10. Let $\mathfrak{R}$ be a ring.

(i) If $x \in \mathfrak{R}$, then $W(x)$ is a compact subset.

(ii) An open subset of $Spec(\mathfrak{R})$ is compact if and only if it is a finite union of sets $W(x_i)$ where $x_i \in \mathfrak{R}$.

Proof. (i) Suppose that $W(x) = \bigcup_{i \in I} W(x_i)$. Then for all prime ideals $\mathcal{P}$ such that $x \notin \mathcal{P}$, there exists i such that $x_i \notin \mathcal{P}$. Let $\mathcal{K}$ be the ideal generated by all x_i. Then if $\mathcal{K} \subset \mathcal{P}$ for some $\mathcal{P}$, we have that $x \in \mathcal{P}$ and hence $x \in \sqrt{\mathcal{K}}$. Therefore there is $m \in \mathbb{N}$ such that $x^m \in \mathcal{K}$, and we write it as $x^m = \sum_{i=1}^{n} a_i x_i$ for some $x_1, x_2, \cdots, x_n$ and $\{a_i\}_{i=1}^{n} \in \mathfrak{R}$. If all $x_1, x_2, \cdots, x_n$ belong to some prime ideal $\mathcal{P}$, we have $x^m \in \mathcal{P}$ and therefore $x \in \mathcal{P}$. Equivalently, $x \notin \mathcal{P}$ for some $\mathcal{P}$ implies $x_i \notin \mathcal{P}$ for some $i = 1, 2, \cdots, n$. Hence, $W(x) = \bigcup_{i=1}^{n} W(x_i)$.

(ii) First, let O be an open set which is a finite union of basic open sets, hence O is compact;(finite union of compact sets is compact). Now, let O be a compact open subset of $Spec(\mathfrak{R})$. So $O = \bigcup_{i \in I} W(x_i)$ and O has a finite subcover, which means there exists $x_1, x_2, \cdots, x_n$ such that $O = \bigcup_{i=1}^{n} W(x_i)$. □

Another topological property of the Zariski topology of a ring $\mathfrak{R}$ is the T_0-space.

Theorem 1.11. Let $\mathfrak{R}$ be a ring. Then $Spec(\mathfrak{R})$ is a T_0-space.

To prove this Theorem we need the following two Lemmas 1.12, 1.13.

Lemma 1.12. Let $\mathcal{A}$ be a nonempty subset of $Spec(\mathfrak{R})$. Then $\overline{\mathcal{A}} = \bigcup_{P \in \mathcal{A}} V(P)$.

Proof. First, since the closure of $\mathcal{A}$ is the smallest closed set containing $\mathcal{A}$ we have $\overline{\mathcal{A}} \subseteq \bigcup_{P \in \mathcal{A}} V(P)$. Now, let $\mathcal{K}$ be any closed subset in $Spec(\mathfrak{R})$ such that $\mathcal{A} \subseteq \mathcal{K}$. Now, there is $\mathcal{S} \subset \mathfrak{R}$ such that $\mathcal{K} = V(\mathcal{S})$. Let Q be a prime ideal such that $Q \in \bigcup_{P \in \mathcal{A}} V(P)$. So there exists $P_0 \in \mathcal{A}$ such that $Q \in V(P_0)$ which means $P_0 \subseteq Q$. Since $\mathcal{A} \subseteq \mathcal{K}$, $P_0 \in \mathcal{K} = V(\mathcal{S})$. Therefore $\mathcal{S} \subseteq P_0 \subseteq Q$, i.e. $Q \in V(\mathcal{S}) = \mathcal{K}$. Hence, $\bigcup_{P \in \mathcal{A}} V(P) \subseteq \mathcal{K}$. But the closure of $\mathcal{A}$ is the intersection of all closed sets containing $\mathcal{A}$, thus $\bigcup_{P \in \mathcal{A}} V(P) \subseteq \overline{\mathcal{A}}$. □

Lemma 1.13. Let $\mathcal{P} \in Spec(\mathfrak{R})$, then $\overline{\{\mathcal{P}\}} = V(\mathcal{P})$.

Proof. It follows by Lemma 1.12, $\overline{\{\mathcal{P}\}} = \bigcup V(\mathcal{P}) = V(\mathcal{P})$ □

Next we prove Theorem 1.11.

Proof. of **Theorem 1.11** Let $\mathcal{P}, \mathcal{Q} \in Spec(\mathfrak{R})$. Then $\overline{\{\mathcal{P}\}} = \overline{\{\mathcal{Q}\}}$ if and only if $V(\mathcal{P}) = V(\mathcal{Q})$, by Lemma 1.13. Now, by Lemma 1.4(iii) $\sqrt{\mathcal{P}} = \sqrt{\mathcal{Q}}$. Now the radical of prime ideals is the ideal itself, so $\mathcal{P} = \mathcal{Q}$. Therefore, by Lemma 0.27, $Spec(\mathfrak{R})$ is a T_0-space. □

In general, $Spec(\mathfrak{R})$ is not a T_1-space under the Zariski topology since singiltons are not always closed. Next we discuss the algebraic properties of $\mathfrak{R}$ that make $Spec(\mathfrak{R})$ a T_1-space. We begin with this Lemma.

Lemma 1.14. Let $\mathcal{P} \in Spec(\mathfrak{R})$. Then $\{\mathcal{P}\}$ is a closed set if and only if $\mathcal{P} \in Max(\mathfrak{R})$.

Proof. Suppose that $\{\mathcal{P}\}$ is closed in $Spec(\mathfrak{R})$. By Lemma 1.13, $\{\mathcal{P}\} = \overline{\{\mathcal{P}\}} = V(\mathcal{P})$. Hence, there is no other prime ideal containing $\mathcal{P}$, which means $\mathcal{P}$ is a maximal ideal. Now, suppose that $\mathcal{P}$ is a maximal ideal of $\mathfrak{R}$. Then by using Lemma 1.13, $\overline{\{\mathcal{P}\}} = V(\mathcal{P}) = \mathcal{P}$ since $\mathcal{P}$ is maximal. Therefore, $\{\mathcal{P}\}$ is a closed subset of $Spec(\mathfrak{R})$. □

Theorem 1.15. Let $\mathfrak{R}$ be a ring. The following assertions are equivalent:

(i) $Spec(\mathfrak{R})$ is a Hausdorff space.

(ii) $Spec(\mathfrak{R})$ is a T_1-space.

(iii) $dim(\mathfrak{R}) = 0$.

Proof. (i)⇒(ii) Trivial (Hausdorff ⇒ T_1-space).

(ii)⇒(iii) Let $Spec(\mathfrak{R})$ be a T_1-space. Then by Lemma 0.28 every singleton $\{p\}$ in $Spec(\mathfrak{R})$ is closed and by Lemma 1.14 every prime ideal of $\mathfrak{R}$ is maximal. This means $dim(\mathfrak{R}) = 0$

(iii)⇒(i) Suppose that $dim(\mathfrak{R}) = 0$. Let $\mathcal{P}$, $\mathcal{Q}$ are two distinct prime (maximal) ideals of $\mathfrak{R}$. Now, assume there exists $f \in \mathcal{P} \setminus \mathcal{Q}$ since $\mathcal{P} \neq \mathcal{Q}$. The local ring $\mathfrak{R}_{\mathcal{P}}$ has exactly one maximal (prime) ideal $\mathcal{P}\mathfrak{R}_{\mathcal{P}}$, which is the nilradical of $\mathfrak{R}_{\mathcal{P}}$. Now, $f/1 \in \mathcal{P}\mathfrak{R}_{\mathcal{P}} = Nil(\mathfrak{R}_{\mathcal{P}})$, i.e. there is $n \geq 1$ such that $(f/1)^n = 0$. This implies that for some $s \in \mathfrak{R} \setminus \mathcal{P}$ we have $sf^n = 0$. Now, $f \notin \mathcal{Q}$ implies that $\mathcal{Q} \in W(f)$ and $s \notin \mathcal{P}$ implies that $\mathcal{P} \in W(s)$. Hence, $W(s) \cap W(f) = \{\mathcal{P}_1 \in Spec(\mathfrak{R}) | s \notin \mathcal{P}_1 \text{ and } f \notin \mathcal{P}_1\} = \{\mathcal{P}_1 \in Spec(\mathfrak{R}) | s \notin \mathcal{P}_1 \text{ and } f^n \notin \mathcal{P}_1\} = W(sf^n) = W(0) = \emptyset$. Thus, $Spec(\mathfrak{R})$ is a Hausdorff space. □

Example 1.16. Let $\mathfrak{R} = \mathbb{Z}$. $Spec(\mathbb{Z})$ is not T_1-space, since (0) is not a maximal ideal in $\mathbb{Z}$.

Recall, $\mathfrak{R}$ is Noetherian if and only if every ascending chain $\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots$ of ideals of $\mathfrak{R}$ is stationary. Also the Noetherian topological space is a space such that it's closed subsets satisfy the descending chain condition. A good point is that the Noetherian property of $\mathfrak{R}$ is highly commuted with the Noetherian property of $Spec(\mathfrak{R})$ under Zariski topology.

Proposition 1.17. Let $\mathfrak{R}$ be a Noetherian ring, then $Spec(\mathfrak{R})$ is a Noetherian space.

Proof. Let $(V(\mathcal{I}_i))_{i\in\mathbb{N}}$ be an infinite collection of closed sets of $Spec(\mathfrak{R})$ such that $V(\mathcal{I}_{i+1}) \subseteq V(\mathcal{I}_i)$. Since $V(\mathcal{I}) = V(\sqrt{\mathcal{I}})$ for any ideal of $\mathfrak{R}$, and by

Lemma 1.3 we have $\sqrt{\mathcal{I}_i} \subseteq \sqrt{\mathcal{I}_{i+1}}$ for all i. Hence we have an ascending chain of ideals of $\mathfrak{R}$ and since $\mathfrak{R}$ is a Noetherian ring there is n such that $\sqrt{\mathcal{I}_i} = \sqrt{\mathcal{I}_n}$ for all $i \geq n$. Therefore, $V(\sqrt{\mathcal{I}_i}) = V(\sqrt{\mathcal{I}_n})$. i.e. $Spec(\mathfrak{R})$ is a Noetherian space. □

From the proof of Proposition 1.17, we notice that $Spec(\mathfrak{R})$ is a Noetherian space if and only if $\mathfrak{R}$ satisfies the a.c.c on the radical ideals.

Example 1.18. [3, Example, page 91] The converse of Proposition 1.17 is not true. Let K be a field and $\mathfrak{R} = K[x_1, x_2, x_3, \cdots]$ be a polynomial ring in a countably infinite set of indeterminates x_i over K, and let the ideal $\mathfrak{a}$ be the ideal $(x_1, x_2^2, x_3^3, \cdots, x_n^n, \cdots)$. The ring $S = \mathfrak{R}/\mathfrak{a}$ has only one prime ideal which is the image of $(x_1, x_2, x_3, \cdots)$. Write $y_n = \overline{x}_n$, so $Spec(S) = (y_1, y_2, y_3, \cdots) = \mathfrak{m}$ is a maximal ideal, since $K[x_1, x_2, x_3, \cdots]/(y_1, y_2, y_3, \cdots) \cong K$ since each $y_n \in Nil(S)$ is a nilpotent, so we have $\mathfrak{m} \subseteq Nil(S)$ so every prime in S is contains $\mathfrak{m}$ which is minimal, hence $\mathfrak{m}$ is the only prime ideal of S . Hence $Spec(S)$ is a finite space so it is a Noetherian space. But S is not a Noetherian ring; the ascending chain $(y_1) \subsetneq (y_1, y_2) \subsetneq (y_1, y_2, y_3) \subsetneq \cdots$ does not terminate.

Recall that, the *discrete topology* on the space T is defined by letting every subset of T be open (and hence also closed). The following Proposition discuss when the Zariski topology on $Spec(\mathfrak{R})$ is the discrete topology.

Proposition 1.19. Let $\mathfrak{R}$ be a Noetherian ring. The following assertions are equivalent:

(1) $\mathfrak{R}$ is Artinian.

(2) $Spec(\mathfrak{R})$ is discrete and finite.

(3) $Spec(\mathfrak{R})$ is discrete.

Proof. $(1) \Rightarrow (2)$: Suppose $\mathfrak{R}$ is Artinian ring. By Proposition 0.11 every prime ideal of $\mathfrak{R}$ is maximal, hence by Lemma 1.14 each point of $Spec(\mathfrak{R})$ is closed. By [3, Proposition 8.3], $\mathfrak{R}$ has a finite number of maximal (prime) ideals, so $Spec(\mathfrak{R})$ is finite. But then every subset of $Spec(\mathfrak{R})$ is a finite union of closed sets, hence closed, and $Spec(\mathfrak{R})$ is discrete.

$(2) \Rightarrow (3)$: Obvious.

$(3) \Rightarrow (1)$: Suppose $Spec(\mathfrak{R})$ is discrete. Then each point is closed. By Lemma 1.14, the closed points correspond to maximal ideals, so every prime ideal in $\mathfrak{R}$ is maximal. Then $dim(\mathfrak{R}) = 0$, so by [3, Proposition 8.5], $\mathfrak{R}$ is Artinian. □

Example 1.20. Let $\mathfrak{R} = \mathbb{Z}_{12}$. First, we find the ideals of $\mathbb{Z}_{12}$. The positive divisors of 12 are $1, 2, 3, 4, 6$ and 12 so we have the following ideals:

$$(1) = \mathbb{Z}_{12}$$

$$(2) = \{0, 2, 4, 6, 8, 10\}$$

$$(3) = \{0, 3, 6, 9\}$$

$$(4) = \{0, 4, 8\}$$

$$(6) = \{0, 6\}$$

$$(12) = \{0\}.$$

Notice that $\mathbb{Z}_{12}$ is both Noetherian and Artinian ring because it is a finite ring so any ascending and descending chain of it's ideals will terminate. Now, the prime ideals of $\mathbb{Z}_{12}$ are (2) and (3) because are maximal ideals. Notice that

$(1), (4), (6)$ and (12) are not prime ideals since $(1), (12)$ are improper ideals while $(4), (6)$ is not prime since $2(2) = 4 \in (4)$ but $2 \notin (4)$ and $2(3) = 6 \in (6)$ but niether 2 nor 3 belongs to (6). Therefore $Spec(\mathbb{Z}_{12}) = \{(2), (3)\}$. Thus $Spec(\mathbb{Z}_{12})$ is discrete space; by using the above Proposition.

Recall that an irreducible topological space is a nonempty space such that every pair of nonempty open sets intersect. The following Proposition can be used to enrich the literature with examples of irreducible spaces.

Proposition 1.21. Let $\mathfrak{R}$ be a ring. Then $Spec(\mathfrak{R})$ is irreducible if and only if $Nil(\mathfrak{R})$ is prime.

Proof. First, suppose that $Spec(\mathfrak{R})$ is irreducible. Let $f, g \notin Nil(\mathfrak{R})$, by Proposition 1.8.(ii), $W(f) \neq \emptyset$ and $W(g) \neq \emptyset$. But $\emptyset \neq W(f) \cap W(g) = W(fg)$ since $Spec(\mathfrak{R})$ is irreducible, hence $fg \notin Nil(\mathfrak{R})$. Therefore, $Nil(\mathfrak{R})$ is prime ideal. Now, suppose $Nil(\mathfrak{R})$ is prime. Let O_1, O_2 be any two nonempty open sets in $Spec(\mathfrak{R})$. There exists $f_1, f_2 \in \mathfrak{R}$ such that $W(f_1) \subseteq O_1$ and $W(f_2) \subseteq O_2$ where $W(f_1), W(f_2)$ are nonempty. Hence by Proposition 1.8.(ii), $f_1, f_2 \notin Nil(\mathfrak{R})$. Now, $W(f_1) \cap W(f_2) = W(f_1 f_2) \neq \emptyset$ since $Nil(\mathfrak{R})$ is prime i.e. $f_1 f_2 \notin Nil(\mathfrak{R})$. Now, $W(f_1) \cap W(f_2) \subseteq O_1 \cap O_2 \neq \emptyset$ since $W(f_1) \cap W(f_2) \neq \emptyset$. Therefore, $Spec(\mathfrak{R})$ is irreducible space. $\square$

Example 1.22. Let $\mathfrak{R}$ be any integral domain. Trivially $\mathfrak{R}$ is reduced ring (i.e. has no nonzero nilpotent). Thus, $Spec(\mathfrak{R})$ is irreducible space.

The following Lemma is useful in finding the irreducible components of $Spec(\mathfrak{R})$.

Lemma 1.23. Let $\mathfrak{R}$ be a ring and $\mathcal{I}$ an ideal of $\mathfrak{R}$. Then $V(\mathcal{I})$ is an irreducible subset of $Spec(\mathfrak{R})$ if and only if $\sqrt{\mathcal{I}}$ is prime.

Proof. Apply Proposition 1.21 to $\mathfrak{R}/\mathcal{I}$. Thus $Spec(\mathfrak{R}/\mathcal{I})$ is irreducible if and only if $Nil(\mathfrak{R}/\mathcal{I})$ is prime. But $Spec(\mathfrak{R}/\mathcal{I}) \cong V(\mathcal{I})$ and $Nil(\mathfrak{R}/\mathcal{I}) = \sqrt{\mathcal{I}}$.

□

Proposition 1.24. Let $\mathfrak{R}$ be a ring. The irreducible components of $Spec(\mathfrak{R})$ are the closed sets $V(\mathcal{P})$, where $\mathcal{P}$ is a minimal prime ideal of $\mathfrak{R}$.

Proof. Suppose Y is a maximal irreducible subset of $Spec(\mathfrak{R})$. Then by Proposition 0.33.(iii), Y is closed. So, $Y = V(P)$ for some ideal P of $\mathfrak{R}$. Now by Lemma 1.23, $\sqrt{P}$ is prime ideal. If there is $Q \in Spec(\mathfrak{R})$ such that $Q \subseteq P$ and $V(Q)$ is irreducible. Then by Lemma 1.4, $V(P) \subseteq V(Q)$. Since $V(P)$ is maximal irreducible subset of $Spec(\mathfrak{R})$, $V(P) = V(Q)$. Hence by Lemma 1.4, $\sqrt{P} = \sqrt{Q}$. Thus $\sqrt{P}$ is a minimal prime ideal. □

1.3 Homeomorphism spectrums

Let $\mathfrak{R}_1$ and $\mathfrak{R}_2$ be two rings and $\phi : \mathfrak{R}_1 \to \mathfrak{R}_2$ be a homomorphism of rings. Recall that, the preimage of a prime ideal of $\mathfrak{R}_2$ is a prime ideal of $\mathfrak{R}_1$. The following Proposition proves that ϕ can be induce a continuous map between $Spec(\mathfrak{R}_2)$ and $Spec(\mathfrak{R}_1)$.

Proposition 1.25. Let $\mathfrak{R}_1, \mathfrak{R}_2$ be two rings and $\phi : \mathfrak{R}_1 \to \mathfrak{R}_2$ be a ring homomorphism. Then $\begin{array}{c} \psi : Spec(\mathfrak{R}_2) \to Spec(\mathfrak{R}_1) \\ \mathcal{P} \to \phi^{-1}(\mathcal{P}) \end{array}$ is a continuous map.

Proof. Let $x \in \mathfrak{R}_1$ and $W(x)$ be an open set in $Spec(\mathfrak{R}_1)$. We want to show that $\psi^{-1}(W(x)) = W(\phi(x))$. Let $\mathcal{P} \in Spec(\mathfrak{R}_2)$ such that $\mathcal{P} \in \psi^{-1}(W(x))$, then $\phi^{-1}(\mathcal{P}) = \psi(\mathcal{P}) \in W(x)$. Now $\phi^{-1}(\mathcal{P})$ is prime in $Spec(\mathfrak{R}_1)$ so $x \notin \phi^{-1}(\mathcal{P})$. Then $\phi(x) \notin \mathcal{P}$ which implies $\mathcal{P} \in W(\phi(x))$ which is open in

$Spec(\mathfrak{R}_2)$. Therefore, $\psi^{-1}(W(x)) \subseteq W(\phi(x))$. In a similar way $W(\phi(x)) \subseteq \psi^{-1}(W(x))$. Therefore, the preimage of open sets in $Spec(\mathfrak{R}_1)$ are open in $Spec(\mathfrak{R}_2)$, i.e. ψ is continuous map. □

The following Proposition proves that the spectrum of a ring $\mathfrak{R}$ is homeomorphic to the spectrum of $\mathfrak{R}/Nil(\mathfrak{R})$.

Proposition 1.26. Let $\mathfrak{R}$ be a ring. Then $Spec(\mathfrak{R}) \overset{\psi}{\cong} Spec(\mathfrak{R}/Nil(\mathfrak{R}))$.

Proof. Define $\phi : \mathfrak{R} \to \mathfrak{R}/Nil(\mathfrak{R})$ by $\phi(\mathcal{I}) = \mathcal{I}/Nil(\mathfrak{R})$, where $\mathcal{I}$ is an ideal of $\mathfrak{R}$. Clearly, $Spec(\mathfrak{R}/Nil(\mathfrak{R})) = \{\mathcal{P}/Nil(\mathfrak{R})|\mathcal{P} \in Spec(\mathfrak{R})\}$. Thus, $\psi : Spec(\mathfrak{R}) \to Spec(\mathfrak{R}/Nil(\mathfrak{R}))$ given by $\psi(\mathcal{P}) = \mathcal{P}/Nil(R)$ is bijection. Now by Proposition 1.25, ψ^{-1} is a continuous map. Finally we show that ψ is continuous. Let $W(\mathcal{J}/Nil(\mathfrak{R}))$ be open in $Spec(\mathfrak{R}/Nil(\mathfrak{R}))$, where $\mathcal{J}$ is an ideal of $\mathfrak{R}$. Suppose $\mathcal{Q} \in \psi^{-1}(W(\mathcal{J}/Nil(\mathfrak{R})))$. Hence, $\psi(\mathcal{Q}) \in W(\mathcal{J}/Nil(\mathfrak{R}))$ so $\mathcal{J}/Nil(\mathfrak{R}) \nsubseteq \psi(\mathcal{Q})$. Then $\mathcal{J} = \psi^{-1}(\mathcal{J}/Nil(\mathfrak{R})) \nsubseteq \mathcal{Q}$ so $\mathcal{Q} \in W(\mathcal{J})$. Therefore, $\psi^{-1}(W(\mathcal{J}/Nil(\mathfrak{R}))) \subseteq W(\mathcal{J})$. In similar argument $W(\mathcal{J}) \subseteq \psi^{-1}(W(\mathcal{J}/Nil(\mathfrak{R})))$. Hence, ψ is a homeomorphism. □

Now the following Proposition discuss the case that $\phi : \mathfrak{R}_1 \to \mathfrak{R}_2$ is a serjective (onto) map.

Proposition 1.27. Let $\mathfrak{R}$ be a ring, $\mathcal{I}$ an ideal of $\mathfrak{R}$. Let $V(\mathcal{I})$ be provided with the subspace topology of $Spec(\mathfrak{R})$. Then $Spec(\mathfrak{R}) \overset{\psi}{\cong} V(\mathcal{I})$.

Proof. Define ϕ as follows
$$\begin{aligned} \phi :&\mathfrak{R} \to \mathfrak{R}/\mathcal{I} \\ &\mathcal{J} \mapsto \overline{\mathcal{J}} := \mathcal{J}/\mathcal{I} \end{aligned}$$

Recall that ϕ is a surjective map and $Ker(\phi) = \mathcal{I}$. Clearly, $Spec(\mathfrak{R}/\mathcal{I}) = \{\mathcal{P}/\mathcal{I}|\mathcal{P} \in Spec(\mathfrak{R})$ and $\mathcal{I} \subseteq \mathcal{P}\}$. Thus, $\psi : Spec(\mathfrak{R}) \to Spec(\mathfrak{R}/\mathcal{I})$ given by

$\psi(\mathcal{P}) = \mathcal{P}/\mathcal{I}$ is a bijection map. Now by Proposition 1.25, ψ^{-1} is a continuous map. Finally we show that ψ is continuous. Suppose $V(\mathcal{J}/\mathcal{I})$ is a closed set in $Spec(\mathfrak{R}/\mathcal{I})$. Then $\psi^{-1}(V(\mathcal{J}/\mathcal{I})) = \{\mathcal{P} \in Spec(\mathfrak{R}) | \mathcal{J}/\mathcal{I} \subseteq \mathcal{P}/\mathcal{I} \in Spec(\mathfrak{R}/\mathcal{I})\} = V(\mathcal{J})$ is a closed set in $Spec(\mathfrak{R})$. Therefore, ψ is a bijection and both ψ and ψ^{-1} are continuous. Thus ψ is homeomorphism. $\square$

Next we consider an example on non homeomorphic map between spectrums of two rings.

Example 1.28. [3, Exercise 21(vii), page 13] Let $\mathfrak{R}$ be an integral domain with one nonzero prime ideal $\mathcal{P}$, i.e. $Spec(\mathfrak{R}) = \{(0), \mathcal{P}\}$ this means $\mathcal{P}$ is a maximal ideal of $\mathfrak{R}$ which implies that $\mathfrak{R}/\mathcal{P}$ is a field. Let K be the field of fractions of $\mathfrak{R}$. Hence the ring $S = (\mathfrak{R}/\mathcal{P}) \times K$ also has exactly two prime ideals, $\mathcal{Q}_1 = \{(\bar{x}, 0) : x \in \mathfrak{R}\}$ and $\mathcal{Q}_2 = \{(\bar{0}, k) : k \in K\}$. Note that, $\mathcal{Q}_1$ is a (maximal) prime ideal since $S/\mathcal{Q}_1 \cong K$ which is a field and $\mathcal{Q}_2$ is a prime ideal since $S/\mathcal{Q}_2 \cong \mathfrak{R}/\mathcal{P}$ which is a field. Define

$$\begin{aligned} \phi :& \mathfrak{R} \to S \\ & x \mapsto (\bar{x}, x). \end{aligned}$$

Notice that $\phi(x + y) = (\overline{x + y}, x + y) = (\overline{x}, x) + (\overline{y}, y) = \phi(x) + \phi(y)$ and $\phi(xy) = (\overline{xy}, xy) = (\overline{x}, x)(\overline{y}, y) = \phi(x)\phi(y)$ for all $x, y \in \mathfrak{R}$ and $\phi(1_{\mathfrak{R}}) = (1_{\mathfrak{R}} + \mathcal{P}, 1_{\mathfrak{R}}) = 1_S$. Thus ϕ is a ring homomorphism. Therefore ϕ induses a map ψ between $Spec(S)$ and $Spec(\mathfrak{R})$ such that $\psi(P) = \phi^{-1}(P)$ for all $P \in Spec(S)$. Hence, $\psi(\mathcal{Q}_1) = \{x \in \mathfrak{R} | \overline{x} = 0\} = \mathcal{P}$ and $\psi(\mathcal{Q}_2) = \{x \in \mathfrak{R} | x = 0\} = (0)$ so ψ is one-to-one and onto. However, ψ is not a homeomorphism since in the topological space $Spec(S) = \{\mathcal{Q}_1, \mathcal{Q}_2\}$, we have

$\mathcal{Q}_1 = V(\mathcal{Q}_1)$ is closed as $\mathcal{Q}_1 \nsubseteq \mathcal{Q}_2$, but $\psi(\mathcal{Q}_2) = (0)$ is not closed in $Spec(\mathfrak{R})$, since (0) is not a maximal ideal of $\mathfrak{R}$, by Lemma 1.14.

1.4 The spectrum of product ring

In this section we discuss some properties of the spectrum of the product of finite number of rings $\{\mathfrak{R}_i\}_{i=1}^n$, denoted by $\prod_{i=1}^{n} \mathfrak{R}_i$ is the set of all $r = (r_1, r_2, \cdots, r_n)$ where $r_i \in \mathfrak{R}_i$ for all $1 \leq i \leq n$. Recall that $\pi_i \colon \prod_{i=1}^{n} \mathfrak{R}_i \to \mathfrak{R}_i$ given by $(r_j) \mapsto r_i$ is the canonical projection map which is a ring homomorphism. The following Theorem talks about the spectrum of direct product of finite number of rings.

Theorem 1.29. Let each $\{\mathfrak{R}_i\}_{i=1}^n$ be a ring. Then

$$Spec(\prod_{i=1}^{n} \mathfrak{R}_i) = \bigcup_{i=1}^{n} X_i,$$

where $X_i \cong Spec(\mathfrak{R}_i)$ and the sets X_i's are disjoint and clopen (i.e closed and open).

Proof. Let $\pi_i \colon \prod_{i=1}^{n} \mathfrak{R}_i \to \mathfrak{R}_i$ given by $(r_j) \mapsto r_i$ be the canonical projection, and $f_i = \mathrm{Ker}(\pi_i) \cong \prod_{j \neq i} \mathfrak{R}_j$. As π_i is surjective, then by Proposition 1.27 the map $\pi_i^* \colon Spec(\mathfrak{R}_i) \to V(f_i) = X_i$ is homeomorphism. Since $\bigcap_{i=1}^{n} f_i = 0$, we have $\bigcup_{i=1}^{n} X_i = V(\bigcap_{i=1}^{n} f_i) = V(0) = Spec(\mathfrak{R})$. Now, for $i \neq j$ we have $V(f_i) \cap V(f_j) = V(f_i \cup f_j) = V(\mathfrak{R}) = \emptyset$. Hence, X_i's are disjoint closed sets which cover $Spec(\mathfrak{R})$. Finally, X_i's are open sets also since the complement of $V(f_i)$ which equal to $\bigcup_{j \neq i} V(f_i)$ is closed. □

The last Theorem above can be used to prove the following result which discuss the connectedness of spectrums of rings under Zariski topology. The following Proposition discuss a characterization of disconnected $Spec(\mathfrak{R})$.

Proposition 1.30. Let $\mathfrak{R}$ be any ring. The following assertions are equivalent:

(i) $\mathfrak{R} \cong \mathfrak{R}_1 \times \mathfrak{R}_2$ where $\mathfrak{R}_1,\ \mathfrak{R}_2$ are non-zero rings.

(ii) $Spec(\mathfrak{R})$ is disconnected.

(iii) $\mathfrak{R}$ contains an idempotent different than 0 and $1_{\mathfrak{R}}$.

Proof. (i)⇒(ii): By Theorem 1.29, $Spec(\mathfrak{R}) = X_1 \bigcup X_2$ where X_1, X_2 are nonempty closed sets and $X_1 \bigcap X_2 = \emptyset$. Hence, $Spec(\mathfrak{R})$ is a disconnected space.

(ii)⇒(iii): Assume that $Spec(\mathfrak{R})$ is disconnected; then it is a disjoint union of two nonempty closed sets $V(\mathcal{I}),\ V(\mathcal{J})$. Then $V(\mathcal{I}) \bigcap V(\mathcal{J}) = V(\mathcal{I}+\mathcal{J}) = \emptyset$, so there is no prime contain $\mathcal{I}+\mathcal{J}$, which must be equal $(1_{\mathfrak{R}})$. Therefore, there exists $x \in \mathcal{I}$ and $y \in \mathcal{J}$ such that $x+y = 1_{\mathfrak{R}}$. Since $Spec(\mathfrak{R})$ is disconnected, we have $Spec(\mathfrak{R}) = V(\mathcal{I}) \bigcup V(\mathcal{J}) = V(\mathcal{I}\mathcal{J})$, so $\mathcal{I}\mathcal{J} \subseteq Nil(\mathfrak{R})$. Thus there is n such that $(xy)^n = 0$. Consider the equation $1_{\mathfrak{R}} = 1_{\mathfrak{R}}^{2n} = (x+y)^{2n} = x^{2n} + \cdots + x^{n+1}y^{n-1} + x^n y^n + x^{n-1}y^{n+1} + \cdots + y^{2n}$, let $a_1 = x^{2n} + \cdots + x^{n+1}y^{n-1}$ and $a_2 = x^{n-1}y^{n+1} + \cdots + y^{2n}$. So $a_1 + a_2 = 1_{\mathfrak{R}}$, and every term in a_1a_2 contains a factor of $(xy)^n$, thus $a_1a_2 = 0$. Therefore $a_1 = a_1(a_1+a_2) = a_1^2 + a_1a_2 = a_1^2$, so a_1 and a_2 are idempotent.

(iii)⇒(i): Suppose $a \neq 0, 1_{\mathfrak{R}}$ is an idempotent. Then $1-a$ is also an idempotent $\neq 0,\ 1_{\mathfrak{R}}$. This means (a) and $(1-a)$ are proper, nonzero ideals, and they are coprime since $a + [1-a] = 1$. Since $(a)(1-a) = (a-a^2) = (0)$, by [3, Proposition 1.10.i] $(a) \cap (1-a) = (a)(1-a) = (0)$. Let $\phi : \mathfrak{R} \to$

$\mathfrak{R}/(a) \times \mathfrak{R}/(1-a)$ given by $\phi(x) = (x + (a), x + (1-a))$ be the natural homomorphism. Then by [3, Proposition 1.10.ii, iii] ϕ is an isomorphism. □

Example 1.31. Let $\mathfrak{R} = \mathbb{Z}_{10}$. First, we find $Spec(\mathbb{Z}_{10})$. The positive divisors of 10 are $1, 2, 5$ and 10 so we have the following ideals:

$$(1) = \mathbb{Z}_{10}$$

$$(2) = \{0, 2, 4, 6, 8\}$$

$$(5) = \{0, 5\}$$

$$(10) = \{0\}.$$

The prime ideals of $\mathbb{Z}_{10}$ are (2) and (5) since both of them are maximal ideals. Notice that (1) and (10) are improper ideals of $\mathbb{Z}_{10}$ so both of them are neither prime nor maximal. Therefore $Spec(\mathbb{Z}_{10}) = \{(2), (5)\}$. Now, notice that $\mathbb{Z}_{10} \cong \mathbb{Z}_2 \times \mathbb{Z}_5$ where neither $\mathbb{Z}_2$ nor $\mathbb{Z}_5$ is the zero ring. Also, the idempotents of $\mathbb{Z}_{10}$ are $0, 1, 5$and 6, i.e. $\mathbb{Z}_{10}$ has an idempotent different than 0 and 1. Therefore, by Proposition 1.30 $Spec(\mathbb{Z}_{10})$ is disconnected space.

Corollary 1.32. The spectrum of a local ring $\mathfrak{R}$ (ring with one maximal ideal) is always connected.

Proof. It follows by Proposition 1.30 since the idempotents of local rings are 0 and $1_{\mathfrak{R}}$; for any idempotent element a, we have $a = a^{-1}a^2 = a^{-1}a = 1$. Suppose $a^2 = a \neq 0, 1_{\mathfrak{R}}$ in $\mathfrak{R}$. Then a is not a unit, and hence it is contained in some maximal ideal $\mathfrak{m}$. Similarly $(1-a)^2 = 1 - 2a + a^2 = 1 - a$ is another idempotent $\neq 0, 1_{\mathfrak{R}}$, hence not a unit. But $\mathfrak{R}$ is local, a would be in $\mathfrak{m} = \bigcap Max(\mathfrak{R})$, so $1 - a$ would be a unit by [3, Proposition 1.9]. □

Example 1.33. (1) Let $\mathfrak{R} = \mathbb{Z}_8$. First, we find $Spec(\mathbb{Z}_8)$. The positive divisors of 8 are $1, 2, 4$ and 8 so we have the following ideals:

$$(1) = \mathbb{Z}_8$$

$$(2) = \{0, 2, 4, 6\}$$

$$(4) = \{0, 4\}$$

$$(8) = \{0\}.$$

The prime ideal of $\mathbb{Z}_8$ is (2) because it is a maximal ideal. Notice that $(1), (4)$ and (8) are not prime ideals since $(1), (8)$ are improper ideals while (4) is not prime since $2(2) = 4 \in (4)$ but $2 \notin (4)$. Therefore $Spec(\mathbb{Z}_8) = \{(2)\}$. Thus $\mathbb{Z}_8$ has one maximal (prime) ideal. Thus $Spec(\mathbb{Z}_8)$ is a connected space; by using the above Corollary. Also notice that the idempotents of $\mathbb{Z}_8$ are 0 and 1.

(2) Let $\mathfrak{R}_p$ be the subset of $\mathbb{Q}$ consisting of rationals $\frac{x}{y}$ where $p \nmid y$ and p is a fixed prime. i.e. $\mathfrak{R}_p = \{\frac{x}{y} | x, y \in \mathbb{Z}, p \nmid y\}$ is a commutative ring. Now, $\mathfrak{R}_p$ has one maximal ideal $\mathcal{I} = \{\frac{x}{y} | x, y \in \mathbb{Z}, p \nmid y, p \mid x\}$. Therefore $\mathfrak{R}_p$ is a local ring, [3, Examples, page 38]. Thus $Spec(\mathfrak{R}_p)$ is a connected space; by using the above Corollary. Also notice that the idempotents of $\mathfrak{R}_p$ are $0_{\mathfrak{R}_p} = 0/1$ and $1_{\mathfrak{R}_p} = 1/1$.

CHAPTER 2

Zariski topology of modules

In this chapter, we will discuss the generalization of Zariski topology of rings to modules which was studied by M. Behboodi and M. R. Haddadi [7, 6].

2.1 Definitions and Basics

In this section, we present the definition of the classical Zariski topology of modules. We start by the following definitions and propositions about the prime spectrum of a module.

Definition 2.1. Let $\mathfrak{R}$ be a ring, E a left $\mathfrak{R}$-module and P a proper submodule of E. Then:

(1) A prime submodule of E is a proper submodule P such that for any ideal $\mathcal{I}$ of $\mathfrak{R}$ and any submodule $\mathcal{N}$ of E, if $\mathcal{I}\,\mathcal{N} \subseteq P$. then $\mathcal{N} \subseteq P$ or $\mathcal{I}\,E \subseteq P$.

(2) A prime module E is a module such that it's zero submodule is prime. Equivalently, for every nonzero submodule $\mathcal{N}$ of E, $Ann_{\mathfrak{R}}(\mathcal{N}) = Ann_{\mathfrak{R}}(E)$.

(3) The set of all prime submodules of E is called the *prime spectrum* of E and denoted by $\mathrm{Spec}(E)$.

Recall that a nonzero left $\mathfrak{R}$-module E is *simple* if it has no submodule except itself and the zero submodule. If E is a direct sum of simple modules then E is called a semisimple module.

Example 2.2. Each simple module is prime. Let $\mathfrak{R} = \mathbb{Z}$ be the ring of integers. The simple $\mathbb{Z}$-modules are of the form $\mathbb{Z}/p\mathbb{Z}$, where p is prime.

The next proposition proves that from a prime submodule of an $\mathfrak{R}$-module E, we can obtain a prime ideal of $\mathfrak{R}$.

Proposition 2.3. Let $\mathfrak{R}$ be a ring, E a left $\mathfrak{R}$-module and P a proper submodule of E . If P is a prime submodule, then $(P : E)$ is a prime ideal of $\mathfrak{R}$.

Proof. [10] Let P be a prime submodule of E. Let $\mathcal{A},\ \mathcal{B}$ be two ideals in $\mathfrak{R}$ such that $\mathcal{A}\,\mathcal{B} \subseteq (P : E)$. Suppose that $\mathcal{B} \nsubseteq (P : E)$, thus there is $b \in \mathcal{B}$ such that $b \notin (P : E)$. Then there is $c \in E$ such that $x = bc \notin P$. Now let a be any element from $\mathcal{A}$. Now $\mathcal{A}\,\mathcal{B} \subseteq (P : E)$ implies $a\mathfrak{R}b \subseteq (P : E)$. Then $(a\mathfrak{R})x = (a\mathfrak{R}b)c \subseteq P$ implies $(\mathfrak{R}a\mathfrak{R})(\mathfrak{R}x) \subseteq P$. Hence $\mathfrak{R}a\mathfrak{R} \subseteq (P : E)$ or $\mathfrak{R}x \subseteq P$, since P is a prime submodule. But $x \notin P$, thus $a \in \mathfrak{R}a\mathfrak{R} \subseteq (P : E)$. Therefore, $\mathcal{A} \subseteq (P : E)$. Finally, $(P : E) \neq \mathfrak{R}$ since P is a proper submodule of E, i.e. $1_{\mathfrak{R}} \notin (P : E)$. □

The next example shows that the converse of the previous proposition is not true in general.

Example 2.4. Let $\mathfrak{R} = \mathbb{Z}$ and $E = \mathbb{Z} \times \mathbb{Z}, \mathcal{N} = (2, 0)\mathbb{Z}$ then $(\mathcal{N} : E) = (0)$ is a prime ideal of $\mathbb{Z}$ but $\mathcal{N}$ is not a prime submodule of $\mathbb{Z} \times \mathbb{Z}$ since $(6)(3, 0)\mathbb{Z} \subseteq (2, 0)\mathbb{Z}$ but $(6)E \nsubseteq \mathcal{N}$ and $(3, 0)\mathbb{Z} \nsubseteq \mathcal{N}$. This is the case for all submodules of the form $(z, 0)\mathbb{Z},\ z > 1$ is an integer.

Remark 2.5. Let $\mathfrak{R}$ be a commutative ring, E a left $\mathfrak{R}$-module and $\mathcal{N}$ a submodule of E. If $(\mathcal{N} : E)$ is a maximal ideal in $\mathfrak{R}$, then $\mathcal{N}$ is a prime submodule of E. For the proof of this remark see [21, Proposition 2].

Clearly the prime submodules of $\mathfrak{R}$ as an $\mathfrak{R}$-module coincide with the prime ideals.

Let $\mathfrak{R}$ be a ring and E an $\mathfrak{R}$-module, throughout for any submodule $\mathcal{N}$ of E, $\boldsymbol{V(\mathcal{N})}$ denotes the set $\boldsymbol{\{P \in Spec(E) \mid \mathcal{N} \subseteq P\}}$, and $\boldsymbol{\mathcal{V}(E)}$ denotes the set $\boldsymbol{\{V(\mathcal{N}) \mid \mathcal{N} \leq E\}}$. Clearly, $V(0) = Spec(E)$, $V(E) = \emptyset$ and for any family of submodules $\mathcal{N}_i$ of E, $\bigcap\limits_{i\in I} V(\mathcal{N}_i) = V(\sum\limits_{i\in I}\mathcal{N}_i)$.

In contrast to the situation of the ring theory, $\mathcal{V}(E)$ is not closed under finite union in general, see the next example.

Example 2.6. Let $\mathfrak{R} = \mathbb{Z}$ and $E = \mathbb{Q} \oplus \mathbb{Q}$. The prime submodules of E are $(0),\ \mathbb{Q} \oplus (0),\ (0) \oplus \mathbb{Q}$ and $\{P_Y \mid \emptyset \neq Y \subseteq \mathbb{Q}^*\}$, where $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$, $P_Y = \bigcup\limits_{t\in Y} P_t$ for every $t \in \mathbb{Q}^*$, $P_t = \{(x, tx) | x \in \mathbb{Q}\}$ see [2, page 4466] for the proof. Let $(0, 1/2),\ (1/2, 3/4) \in E$, $V((0, 1/2)) \cup V(1/2, 3/4) = (0) \oplus \mathbb{Q} \cup P_{3/2} \subseteq V(0, 0)$. But $V((0, 0)) = Spec(E)$. And there is no submodule $\mathcal{N}$ of E such that $V(\mathcal{N}) = (0) \oplus \mathbb{Q} \cup P_{3/2}$.

The previous discussion motivate to introduce the following definition.

Definition 2.7. [26] Let $\mathfrak{R}$ be a ring, E an $\mathfrak{R}$-module, E is called a module with Zariski topology or *top module* if for any submodules $\mathcal{N}, \mathcal{M}$ of E there exists a submodule $\mathcal{L}$ such that $V(\mathcal{N}) \cup V(\mathcal{M}) = V(\mathcal{L})$.

Example 2.8. Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module.

(1) If $Spec(E)$ is empty, then E is a top module. e.g. Let $\mathfrak{R} = \mathbb{Z}$, p a fixed prime and
$0 \neq E = E(p) = \{\alpha \in \mathbb{Q}/\mathbb{Z} \big| \alpha = r/p^n + \mathbb{Z}$ for some $r \in \mathbb{Z}$ and $n \in \mathbb{N}\}$
see [18] for the proof.

(2) If E is a module that has only one prime submodule, then it is a top module. e.g. $\mathbb{Q}$ as a $\mathbb{Z}$-module by using [18, Theorem 1], $Spec(\mathbb{Q}) = \{(0)\}$ and $Max(\mathbb{Q}) = \emptyset$.

(3) If $E = \mathfrak{R}$, then it is a top module. This topological space was studied in chapter one of this thesis.

Recall that an $\mathfrak{R}$-module E is called a *multiplication module* if, for each submodule $\mathcal{N}$ of E, there exists an ideal $\mathcal{I}$ of $\mathfrak{R}$ such that $\mathcal{N} = \mathcal{I}\, E$.
The following proposition is needed to prove that the multiplication modules are top modules.

Proposition 2.9. ([26, Lemma 3.1]) Let $\mathfrak{R}$ be a ring, E a left $\mathfrak{R}$-module, $\mathcal{I}$ an ideal of $\mathfrak{R}$ and $\mathcal{N}$ a submodule of E. Then

$$V(\mathcal{N}) \cup V(\mathcal{I}E) = V(\mathcal{I}\mathcal{N}).$$

Proof. First, let $\mathcal{P}$ be a prime submodule such that $\mathcal{P} \in V(\mathcal{N}) \cup V(\mathcal{I}E)$. Then $\mathcal{P} \in V(\mathcal{N})$ or $\mathcal{P} \in V(\mathcal{I}E)$. Thus $\mathcal{N} \subseteq \mathcal{P}$ or $\mathcal{I}E \subseteq \mathcal{P}$ but $\mathcal{N} \subseteq E$ so $\mathcal{I}\mathcal{N} \subseteq \mathcal{P}$, i.e. $\mathcal{P} \in V(\mathcal{I}\mathcal{N})$. Thus, we proved that $V(\mathcal{N}) \cup V(\mathcal{I}E) \subseteq V(\mathcal{I}\mathcal{N})$, Now, let $\mathcal{P} \in V(\mathcal{I}\mathcal{N})$ be a prime submodule. Then $\mathcal{I}\mathcal{N} \subseteq \mathcal{P}$, so $\mathcal{N} \subseteq \mathcal{P}$ or $\mathcal{I}E \subseteq \mathcal{P}$. Thus $\mathcal{P} \in V(\mathcal{N})$ or $\mathcal{P} \in V(\mathcal{I}E)$, i.e. $\mathcal{P} \in V(\mathcal{N}) \cup V(\mathcal{I}E)$. Thus, we proved that $V(\mathcal{I}\mathcal{N}) \subseteq V(\mathcal{N}) \cup V(\mathcal{I}E)$ □

Corollary 2.10. ([26, Corollary 3.2]) Let $\mathfrak{R}$ be a ring, E an $\mathfrak{R}$-module and $\mathcal{I},\ \mathcal{J}$ ideals of $\mathfrak{R}$. Then $V(\mathcal{I}E) \cup V(\mathcal{J}E) = V(\mathcal{I}\mathcal{J}E)$.

Throughout let $\mathfrak{R}$ be a ring, E a left $\mathfrak{R}$-module and $\mathcal{N}$ a submodule of E. Then $\mathbf{W}(\mathcal{N})$ denotes the set $\mathbf{Spec}(\mathbf{E}) \setminus \mathbf{V}(\mathcal{N}) = \{\mathbf{P} \in \mathbf{Spec}(\mathbf{E}) | \mathcal{N} \nsubseteq \mathbf{P}\}$, and $\mathcal{W}(\mathbf{E})$ denotes the set $\{\mathbf{W}(\mathcal{N}) \mid \mathcal{N} \leq \mathbf{E}\}$.

Now we present the generalization of the Zariski topology of rings to modules, which was called the classical Zariski topology of modules and was introduced by M. Behboodi and M. R. Haddadi in 2008.

Definition 2.11. Let $\mathfrak{R}$ be a ring and E be an $\mathfrak{R}$-module. Then $\mathcal{T}(E)$ is the collection of all unions of finite intersections of elements of $\mathcal{W}(E)$. This collection is the topology on $Spec(E)$ by the subbasis $\mathcal{W}(E)$ and it is called the *classical Zariski topology of* E.

Definition 2.12. The set U is an open subset in this topology if it is a unions of finite intersections of elements of $\mathcal{W}(E)$.

Remark 2.13. Let $\mathfrak{R}$ be a ring.

- The classical Zariski topology of the module $\mathfrak{R}$ as $\mathfrak{R}$-module and the usual Zariski topology of the ring $\mathfrak{R}$ are coincide.
- If E is a top $\mathfrak{R}$-module, then the classical Zariski topology and the Zariski topology of E are coincide.

2.2 Properties of classical Zariski topology

We begin this section by recalling some consepts that will emerge during the rest of this chapter.

Definition 2.14. ([5]) Let $\mathfrak{R}$ be a ring, E a left $\mathfrak{R}$-module and $\mathcal{P}$ a proper submodule of E.

- A *maximal prime* submodule $\mathcal{P}$ of E is a prime submodule of E that does not contain in another prime submodule.

- E is a *homogeneous semisimple* module if it is a direct sum of isomorphic simple $\mathfrak{R}$-modules. Equivalently, $Ann_{\mathfrak{R}}(E)$ is a maximal ideal of $\mathfrak{R}$.

- If the factor module $E/\mathcal{P}$ is a homogeneous semisimple, then $\mathcal{P}$ is called a *virtually maximal* submodule of E.

The following Theorem provides an algebraic characterization for $Spec(E)$ to be a T_1-space.

Lemma 2.15. Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. If $Spec(E)$ is a T_1-space, then every prime submodule of E is maximal prime.

Proof. Let $\mathfrak{q}_1 \in Spec(E)$, since singleton sets are closed in the T_1-spaces, then $\{\mathfrak{q}_1\}$ is closed set in $Spec(E)$. Hence, $\{\mathfrak{q}_1\} = \bigcap_{i\in I} O_i$, where $O_i = \bigcup_{j=1}^{k_i} V(K_{i,j_i})$, $K_{i,j_i} \leq E$ and I is an index set. Assume there exists a prime submodule $\mathfrak{q}_2$ of E such that $\mathfrak{q}_1 \subsetneqq \mathfrak{q}_2$. Then for any $i \in I$, $\mathfrak{q}_1 \in O_i$, so there exists K_{i,j_i} for some $1 \leq j_i \leq k_i$ such that $\mathfrak{q}_1 \in V(K_{i,j_i})$. Thus $K_{i,j_i} \subseteq \mathfrak{q}_1 \subsetneqq \mathfrak{q}_2$. Now $\mathfrak{q}_2 \in V(K_{i,j_i})$, so $\mathfrak{q}_2 \in O_i$ for all $i \in I$. Therefore, $\mathfrak{q}_2 \in \{\mathfrak{q}_1\}$, which contradicts the assumption. □

Before the following Theorem, we recall a needed definition.

Definition 2.16. [24, Definition 3.1] Let $\mathfrak{R}$ be a ring and E an $\mathfrak{R}$-module. A prime chain of length n is a chain $\mathcal{P}_0 \subset \mathcal{P}_1 \cdots \subset \mathcal{P}_n$ of proper inclusions of prime submodules of E, the prime dimension of E, $dim(E)$, is the maximal length of a prime chain or ∞ if there are prime chains of unbounded length. If E has no prime submodule, we set $dim(E) = -1$.

Theorem 2.17. [7, Theorem 2.14] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module, $Spec(E)$ is a T_1-space if and only if $dim(E) \leq 0$.

Proof. First notice that if $Spec(E) = \emptyset$, then $dim(E) = -1$ and trivially it is a T_1-space. So we may assume that $Spec(E) \neq \emptyset$. Suppose $Spec(E)$ is a T_1-space, then by the previous Lemma, if $\mathcal{P}$ is a prime submodule of E, then there is no other prime containing $\mathcal{P}$. Thus $dim(E) = 0$. Now suppose $dim(E) = 0$; if $\mathcal{P}$ is a prime submodule of E, then it is a maximal prime submodule. Thus $V(\mathcal{P}) = \{\mathcal{P}\}$ for all $\mathcal{P} \in Spec(E)$, and so $\{\mathcal{P}\}$ is a closed set in $Spec(E)$. Therefore, $Spec(E)$ is a T_1-space. □

The following corollary insure that Theorem 2.17 recovers Theorem 1.15.

Corollary 2.18. Let $\mathfrak{R}$ be a commutative ring and $E = \mathfrak{R}$. Then $Spec(\mathfrak{R})$ is a T_1-space if and only if $dim(\mathfrak{R}) = 0$.

Corollary 2.19. [7, Lemma 2.32] Let E be a prime module, $Spec(E)$ is a T_1-space if and only if $Spec(E) = \{(0)\}$.

Proof. It follows since E is prime means the zero submodule is prime submodule and then use Theorem 2.17. □

For a finitely generated module E, if $\mathcal{Q}$ is a proper submodule of E, then $\mathcal{Q}$ is contained in a maximal submodule,by [12, Corollary 10.5]. Hence by Theorem 2.17, $Spec(E)$ is a T_1-space if and only if $Spec(E) = Max(E)$. Next result treats the case when the module is finitely generated.

Theorem 2.20. [7, Theorem 2.17] Let $\mathfrak{R}$ be a commutative ring and E a finitely generated $\mathfrak{R}$-module, $Spec(E)$ is a T_1-space if and only if E is multiplication module with $dim(E) = 0$.

Proof. $\Leftarrow$ follows directly by Theorem 2.17.

$\Rightarrow$ Suppose that $Spec(E)$ is a T_1-space. Since E is finitely generated, if $\mathcal{P}$ is a proper submodule of E, then $\mathcal{P}$ is contained in a maximal submodule, by Theorem 2.17, $dim(E) = 0$. Thus by [32, Corollary 4.15], E is a multiplication module. □

Recall that, the cofinite topology is a topology that can be defined on every set T. It has the empty set and all cofinite subsets of T as open sets. Also, the cofinite topology is the smallest topology satisfying the T_1 axiom. Next theorem will discuss when a module E has the cofinite topology on $Spec(E)$.

Theorem 2.21. [7, Theorem 2.22] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. Then $Spec(E)$ is the cofinite topology if and only if $dim(E) \leq 0$ and if $\mathcal{N}$ is a submodule of E then either $V(\mathcal{N}) = Spec(E)$ or $V(\mathcal{N})$ is finite.

Proof. $\Rightarrow$ Assume that $Spec(E)$ is the cofinite topology. By Theorem 2.17, $dim(E) \leq 0$ since $Spec(E)$ is a T_1-space. Now, if there is a submodule $\mathcal{N}$ of E such that $V(\mathcal{N})$ is infinite and $V(\mathcal{N}) \neq Spec(E)$. Then $W(\mathcal{N})$ is an open set in $Spec(E)$ with infinite complement, a contradiction.

$\Leftarrow$ Suppose $dim(E) \leq 0$ and $V(\mathcal{N})$ is finite or $V(\mathcal{N}) = Spec(E)$ for all submodule $\mathcal{N}$ of E. Thus, if $\mathcal{N}_i$'s are submodules of E, then the finite union $\bigcup_{i=1}^{n} V(\mathcal{N}_i)$ is finite or $Spec(E)$. Therefore, any intersection of finite union of submodules $\mathcal{N}_{ji}$ of E, $\bigcap_{j\in J}(\bigcup_{i=1}^{n} V(\mathcal{N}_{ji}))$, is finite or $Spec(E)$. Hence, $Spec(E)$ is the cofinite topology because every closed set in $Spec(E)$ is either finite or $Spec(E)$. □

The following Corollary discuss when $\mathfrak{R}$ as an $\mathfrak{R}$-module is the cofinite topology.

Corollary 2.22. Let $\mathfrak{R}$ be a commutative ring. Then the following assertions are equivalent:

(1) $Spec(\mathfrak{R})$ is the cofinite topology.

(2) $dim(\mathfrak{R}) = 0$ and if $\mathcal{I}$ is a ideal of $\mathfrak{R}$ then either $V(\mathcal{I}) = Spec(\mathfrak{R})$ or $V(\mathcal{I})$ is finite.

Equivalently, $Spec(\mathfrak{R})$ is the cofinite topology if and only if $dim(\mathfrak{R}) = 0$ and $\mathcal{I} \subseteq Nil(\mathfrak{R})$ or $V(\mathcal{I})$ is finite, for any ideal $\mathcal{I}$ of $\mathfrak{R}$.
The following example shows that the condition in Theorem 2.21, $dim(E) = 0$, is not enough to give a cofinite topology on $Spec(E)$. Note that if $dim(E) = -1$, $Spec(E)$ is the trivial space, hence it is cofinite.

Example 2.23. [7, Example2.23] Let $E = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \cdots \oplus \mathbb{Z}_{p_i} \oplus \cdots$ be a $\mathbb{Z}$-module where p_i is a non-negative prime number. Now let $\mathcal{P}$ be a prime submodule of E. Since E is a semisimple module (direct sum of simple $\mathbb{Z}$-modules), $\mathcal{P}$ is virtually maximal by [5, Proposition 1.4]. Hence by Definition 2.14, $E/\mathcal{P}$ is homogeneous semisimple. Since E is a direct sum of non-isomorphic simple modules, $E/\mathcal{P}$ is simple since there is no submodule other than (0) and $E/\mathcal{P}$. So, $\mathcal{P}$ is a maximal submodule of E. Thus $dim(E) = 0$. Clearly for each prime number p_j, $\mathcal{P}_j := \sum_{p_i \neq p_j} \mathbb{Z}_{p_i}$ is a maximal submodule of E and so $Spec(E)$ is infinite. Now, $\mathcal{N} = 0 \oplus \mathbb{Z}_3 \oplus 0 \oplus 0 \oplus \cdots$ is a submodule of E, $V(\mathcal{N}) = Spec(E) \setminus \{\mathcal{P}_2\}$. Thus $V(\mathcal{N})$ is infinite and $V(\mathcal{N}) \neq Spec(E)$. Thus,$Spec(E)$ is not the cofinite topology by Theorem 2.21.

2.3 Classical Zariski topology and Hausdorff spaces

In this section, we discuss when $Spec(E)$ is a Hausdorff space. We start this section by showing when the Artinian module satisfies the ascending chain condition (a.c.c) on intersections of prime submodules.

Definition 2.24. Let $\mathfrak{R}$ be a ring.

- The prime radical $rad_{\mathfrak{R}}(E)$ of a left $\mathfrak{R}$-module E is

$$rad_{\mathfrak{R}}(E) = \bigcap_{P \in Spec(E)} P.$$

- If $Spec(E) = \emptyset$, then $rad_{\mathfrak{R}}(E) = E$.

By a (left) primitive ideal we mean an ideal which is the annihilator of a (nonzero) simple left module. A left primitive ring is a ring which has a faithful (it's annihilator is zero) simple left module. Actully, the quotient of a ring by a left primitive ideal is a left primitive ring [15].

Lemma 2.25. ([30, Theorem 1.5]) Let $\mathfrak{R}$ be a ring such that $\mathfrak{R}/\mathfrak{p}$ is (left) Artinian, where $\mathfrak{p}$ is any primitive ideal of $\mathfrak{R}$ and E is an Artinian $\mathfrak{R}$-module. Then E satisfies a.c.c on intersection of prime submodules.

Proof. Let $\mathfrak{R}$ be a ring such that $\mathfrak{R}/\mathfrak{p}$ is (left) Artinian, where $\mathfrak{p}$ is any primitive ideal. If $Spec(E) = \emptyset$ then the result is true. Suppose that E has a prime submodule. Let $\phi := \{N \leq E | N = \bigcap_{i=1}^{n} P_i, P_i \text{ prime submodule of } E\}$. Then ϕ has a minimal element, say L, because E is Artinian module. Cleary $L = P_1 \cap P_2 \cap \cdots \cap P_n$ where P_i is a prime submodule of E, for all $i = \{1, 2, .., n\}$. Now, let $\mathcal{P}$ be any prime submodule of E, we have $\mathcal{P} \cap P_1 \cap P_2 \cap \cdots \cap P_n \subseteq$

$P_1 \cap P_2 \cap \cdots \cap P_n = L$. Since L is minimal, $\mathcal{P} \cap P_1 \cap P_2 \cap \cdots \cap P_n = L$. Hence $L \leq \mathcal{P}$, so L is contained in any intersection of prime submodules. Consider, P_1. Now $P_1 \neq E$ and hence there is a submodule U of Artinian module E containing P_1 such that U/P_1 is simple. Hence $\mathfrak{p} = Ann(U/P_1)$ is a primitive ideal, also $\mathfrak{p}$ is a prime ideal since P_1 is prime submodule. By our assumption, the ring $\mathfrak{R}/\mathfrak{p}$ is simple Artinian. Hence by [8, Corollary 1.9.] E/P_1 is semisimple since E/P_1 is Artinian and prime module. Thus E/P_i is semisimple, for all $i = \{1, 2, .., n\}$. Thus, E/P_i is Neotherian for all $i = \{1, 2, .., n\}$, since it is Artinian. Hence, E/L is Neotherian, so it satisfies a.c.c on intersection of prime submodules. Hence, E satisfies a.c.c on intersection of prime submodules. □

Corollary 2.26. [7, Proposition 2.29] Let $\mathfrak{R}$ be a ring such that $\mathfrak{R}/\mathfrak{p}$ is (left) Artinian for all primitive ideal $\mathfrak{p}$, and E is an Artinian $\mathfrak{R}$-module. Then $E/rad_{\mathfrak{R}}(E)$ is Noetherian and E satisfies a.c.c on intersections of prime submodules.

Proof. By the proof of Lemma 2.25, if $Spec(E) = \emptyset$, then $rad_{\mathfrak{R}}(E) = E$. Now, if $Spec(E) \neq \emptyset$, then E/P is Noetherian for any prime submodule P. Hence, the radical of E/P is finite intersections of prime submodules, thus $rad_{\mathfrak{R}}(E)$ is finite intersection of prime submodules. Also, $E/rad_{\mathfrak{R}}(E)$ is Noetherian and E satisfies a.c.c on intersections of prime submodules. □

Definition 2.27. Let $\mathfrak{R}$ be a ring.

- A *PI-ring* (polynomial identity ring) is a ring such that all it's elements satisfy a monic polynomial in $\mathbb{Z}[x_1, x_2, \cdots]$, e.g. commutative rings are PI-ring since it's elements satisfy $f(x, y) = xy - yx \in \mathbb{Z}[x, y]$,see [28, Chapter 13].

- A *prime ring* $\mathfrak{R}$ is a ring such that for any two ideals $\mathcal{I},\ \mathcal{J}$ of $\mathfrak{R}$ if $\mathcal{I}\,\mathcal{J} = (0)$, then $\mathcal{I} = (0)$ or $\mathcal{J} = (0)$, e.g. simple rings are prime. Also, integral domains are prime rings.

- If $\mathfrak{R}$ is a prime ring , then $\mathfrak{R}$ is *left bounded* if, for any nonzero divisor c in $\mathfrak{R}$, there is an ideal $\mathcal{A}$ of $\mathfrak{R}$ and a nonzero divisor element d such that $\mathfrak{R}d \subseteq \mathcal{A} \subseteq \mathfrak{R}c$. Recall that, c is a *nonzero divisor* in $\mathfrak{R}$ if there is no $a \in \mathfrak{R}$ such that $ac = 0$.

- A ring $\mathfrak{R}$ is *left fully bounded* if all prime homomorphic image of $\mathfrak{R}$ ($\mathfrak{R}/P$ where P is a prime ideal of $\mathfrak{R}$) is left bounded.

- A ring $\mathfrak{R}$ is *left* FBN-*ring* if $\mathfrak{R}$ is left fully bounded and left Noetherian, see [13] for examples.

Finally, [13, Proposition 9.4] proved that the primitive homomorphic image of an FBN-ring is simple Artinian.

Corollary 2.28. [7, Corollary 2.30] Let $\mathfrak{R}$ be a PI-ring (or an FBN-ring). If E is an Artinian $\mathfrak{R}$-module, then $E/rad_{\mathfrak{R}}(E)$ is Noetherian and E satisfies a.c.c on intersections of prime submodules.

Proof. Every left primitive image of PI-ring is Artinian by [30], thus the result follows by Corollary 2.26. □

Recall that, the topological space T is a *Hausdorff space* if any two distinct points of T can be separated by disjoint open subsets. Hausdorff spaces are also called T_2-spaces or separated spaces. The *discrete topology* on the set X is defined by letting every subset of X be open (and hence also closed). Every discrete topological space satisfies each of the separation axioms. If X is finite, then X is a T_1-space if and only if it is the discrete topology.

By using the previous results we get the following Proposition.

Proposition 2.29. [7, ,Corollary 2.27, Theorem 2.31] Let $\mathfrak{R}$ be a ring and E an $\mathfrak{R}$-module. If either $Spec(E)$ is finite or if $\mathfrak{R}$ is a PI-ring (or an FBN-ring) and E an Artinian module. Then the following assertions are equivalent:

(i) $Spec(E)$ is a Hausdorff space.

(ii) $Spec(E)$ is a T_1-space.

(iii) $Spec(E)$ is the cofinite topology.

(iv) $Spec(E)$ is discrete .

(v) $dim(E) \leq 0$.

Proof. If $\mathfrak{R}$ is a PI-ring and E is an Artinian module, then $E/rad_{\mathfrak{R}}(E)$ is Noetherian and by [25, Theorem 4.2], $Spec(E/rad_{\mathfrak{R}}(E))$ is finite. Now there is a bijective map

$$\mathfrak{f}\colon Spec(E) \to Spec(E/rad_{\mathfrak{R}}(E))$$
$$P \to P + rad_{\mathfrak{R}}(E)$$

(i)⇒(ii) Clear, since every Hausdorff space is a T_1-space.

(ii)⇒(iii) If $Spec(E)$ is a T_1-space, then by Theorem 2.17, $dim(E) \leq 0$. Then $V(N)$ is finite for every $N \leq E$ because $Spec(E)$ is finite. Thus by Theorem 2.21, $Spec(E)$ is a cofinite topology.

(iii)⇒(iv) Suppose that $Spec(E)$ is the cofinite topology. Since singleton sets are closed, $Spec(E)$ is a T_1-space. Thus $Spec(E)$ is discrete, since it is finite.

(iv)⇒(v) Suppose that $Spec(E)$ is discrete. Since $Spec(E)$ is a T_1-space, $dim(E) \leq 0$ by Theorem 2.17.

(v)⇒(i) Suppose that $dim(E) \leq 0$. By Theorem 2.17, $Spec(E)$ is a T_1-space. Since $Spec(E)$ is finite, it is a discrete space. Thus $Spec(E)$ is a Hausdorff space. □

In the next Corollary we apply the above Proposition to commutative ring $\mathfrak{R}$ as an $\mathfrak{R}$-module. Recall that any nonzero commutative ring has one maximal (prime) ideal at least.

Corollary 2.30. Let $\mathfrak{R}$ be a commutative ring such that $Spec(\mathfrak{R})$ is finite. Then the following assertions are equivalent

(i) $Spec(\mathfrak{R})$ is a Hausdorff space.

(ii) $Spec(\mathfrak{R})$ is a T_1-space.

(iii) $Spec(\mathfrak{R})$ is the cofinite topology.

(iv) $Spec(\mathfrak{R})$ is discrete .

(v) $dim(\mathfrak{R}) = 0$.

Next, let $\mathfrak{R}$ be a ring and E an $\mathfrak{R}$-module, we discuss the property of $Spec(E)$ when it is a Hausdorff space. Note that, if $Spec(E)$ is empty or contain one prime submodule then $Spec(E)$ is a Housdorff space since it is the trivial space.

Proposition 2.31. [7, Proposition 2.26] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module such that E has more than two prime submodules. If $Spec(E)$ is a Hausdorff space, then every prime submodule is maximal prime and $Spec(E)$ is covered by finite closed sets $V(\mathcal{N}_i) \neq Spec(E)$ where $\{\mathcal{N}_i\}_{i=1}^{n}$ are submodules of E.

Proof. Suppose that $Spec(E)$ is a Hausdorff space. Let $\mathcal{P}, \mathcal{Q}$ be distinct prime submodules of E. Then there exists disjoint open sets U, V such that $\mathcal{P} \in U$

and $\mathcal{Q} \in V$, where $U = \bigcup_{i\in I}(\bigcap_{j=1}^{n_i} W(\mathcal{N}_{ij}))$, $V = \bigcup_{k\in K}(\bigcap_{l=1}^{m_l} W(\mathcal{N}'_{kl}))$ for submodules $\mathcal{N}_{ij},\ \mathcal{N}'_{kl}$ of E. Then there exists $s \in I,\ t \in K$ such that $\mathcal{P} \in \bigcap_{j=1}^{n_s} W(\mathcal{N}_{sj})$ and $\mathcal{Q} \in \bigcap_{l=1}^{m_t} W(\mathcal{N}_{tl})$, where

$$\left[\bigcap_{j=1}^{n_s} W(\mathcal{N}_{sj})\right] \bigcap \left[\bigcap_{l=1}^{m_t} W(\mathcal{N}_{tl})\right] = \emptyset. \qquad (*)$$

Hence $\mathcal{P} \nsubseteq \mathcal{Q}$ and $\mathcal{Q} \nsubseteq \mathcal{P}$. Therefore, every prime submodule of E is maximal prime. Now by taking the complement for both sides of (*) we get
$\left[\bigcup_{j=1}^{n_s} V(\mathcal{N}_{sj})\right] \bigcup \left[\bigcup_{l=1}^{m_t} V(\mathcal{N}_{tl})\right] = Spec(E).$ □

2.4 Irreducible subsests in classical Zariski topology of modules

Let $\mathfrak{R}$ be a ring, E be a left $\mathfrak{R}$-module and $Spec(E)$ be provided with the classical Zariski topology. In this section we discuss when a subset of $Spec(E)$ is an irreducible subset. We begin this section by recalling that, for any subset A in $Spec(E)$, the *closure* of A is the smallest closed set containing A and is denoted by $\overline{A}$. The following Proposition describe the closure of subsets of $Spec(E)$.

Proposition 2.32. [7, Proposition 3.1] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. If $\mathcal{A} \neq \emptyset$ is a subset of $Spec(E)$, then $\overline{\mathcal{A}} = \bigcup_{\mathcal{P}\in\mathcal{A}} V(\mathcal{P})$.

Proof. The closure of $\mathcal{A}$ is the smallest closed set containing $\mathcal{A}$ thus $\overline{\mathcal{A}} \subseteq \bigcup_{\mathcal{P}\in\mathcal{A}} V(\mathcal{P})$. Let $\mathcal{B}$ be any closed subset of $Spec(E)$ such that $\mathcal{A} \subseteq \mathcal{B}$. Thus $\mathcal{B} = \bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(\mathcal{N}_{ij}))$, for some $\mathcal{N}_{ij}$ submodules of $E,\ i \in I$ and $n_i \in \mathbb{N}$. Now, let $\mathcal{Q} \in \bigcup_{\mathcal{P}\in\mathcal{A}} V(\mathcal{P})$. Then there exists $\mathcal{P}_0 \in \mathcal{A}$ such that $\mathcal{Q} \in V(\mathcal{P}_0)$, so

$\mathcal{P}_0 \subseteq \mathcal{Q}$. Since $\mathcal{A} \subseteq \mathcal{B}$, $\mathcal{P}_0 \in \mathcal{B}$, so for each $i \in I$ there is $j \in \{1, 2, \cdots n_i\}$ such that $\mathcal{N}_{ij} \subseteq \mathcal{P}_0$. Thus $\mathcal{N}_{ij} \subseteq \mathcal{P}_0 \subseteq \mathcal{Q}$. Hence $\mathcal{Q} \in \mathcal{B}$. Therefore $\bigcup_{\mathcal{P}\in\mathcal{A}} V(\mathcal{P}) \subseteq \mathcal{B}$. Finally, since the closure of $\mathcal{A}$ is the intersection of all closed subsets containing it, $\bigcup_{\mathcal{P}\in\mathcal{A}} V(\mathcal{P}) \subseteq \overline{\mathcal{A}}$. □

The following Corollary discuss some properties of the closed set $V(\mathcal{P})$ where $\mathcal{P}$ is a prime submodule.

Corollary 2.33. [7, Corollary 3.2] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. Then

(i) For every $\mathcal{P} \in Spec(E)$, $\overline{\{\mathcal{P}\}} = V(\mathcal{P})$.

(ii) Let $\mathcal{P} \in Spec(E)$. The singleton $\{\mathcal{P}\}$ is closed in $Spec(E)$ if and only if $\mathcal{P}$ is a maximal prime submodule of E.

Proof. By Proposition 2.32 (i) $\overline{\{\mathcal{P}\}} = \bigcup_{F\in\{\mathcal{P}\}} V(F) = V(\mathcal{P})$. Next, for (ii) the set $\{\mathcal{P}\}$ is closed if and only if $\{\mathcal{P}\} = \overline{\{\mathcal{P}\}} = V(\mathcal{P})$ i.e $\mathcal{P}$ is the only prime submodule containing $\mathcal{P}$, thus $\mathcal{P}$ is a maximal prime submodule. □

Now, we start discussing the irreducible subsets of $Spec(E)$ under classical Zariski topology. The following Lemma prove that $V(\mathcal{P})$ is an irreducible subset. Recall that, the *irreducible space* T is nonempty and cannot be expressed as a union of two proper closed subsets, i.e. If $T \subseteq U \cup V$ where U, V are closed subsets of T, then $T \subseteq U$ or $T \subseteq V$.

Lemma 2.34. [7, Lemma 3.3] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. Then $V(\mathcal{P})$ is irreducible for each $\mathcal{P} \in Spec(E)$.

Proof. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two closed sets such that $V(\mathcal{P}) \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$. Now, either $\mathcal{P} \in \mathcal{A}_1$ or $\mathcal{P} \in \mathcal{A}_2$ because $\mathcal{P} \in V(\mathcal{P})$. Say $\mathcal{P} \in \mathcal{A}_1$. Now, $\mathcal{A}_1 =$

$\bigcap_{i\in I}(\bigcup_{j=1}^{n_i} V(\mathcal{N}_{ij}))$, for some index set I, $n_i \in \mathbb{N}$ and $\mathcal{N}_{ij}$ are submodules of E, thus $\mathcal{P} \in \bigcup_{j=1}^{n_i} V(\mathcal{N}_{ij})$, for all $i \in I$. Now $\bigcup_{j=1}^{n_i} V(\mathcal{N}_{ij})$ is closed so it contains $\overline{\{\mathcal{P}\}}$. Thus $V(\mathcal{P}) \in \bigcup_{j=1}^{n_i} V(N_{ij})$, for all $i \in I$. Therefore, $V(\mathcal{P}) \subseteq \mathcal{A}_1$. i.e. $V(\mathcal{P})$ is irreducible. □

For any subset A in $Spec(E)$, the intersection of elements of A is denoted by $\Im(A)$ (note that if $A = \emptyset$, then $\Im(A) = E$). The next Theorem shows that we can obtain a prime submodule of E from an irreducible subset of $Spec(E)$.

Theorem 2.35. [7, Theorem 3.4] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. If $\mathcal{A}$ is an irreducible subset of $Spec(E)$, then $\Im(\mathcal{A})$ is a prime submodule of E.

Proof. Let $\mathcal{A}$ be an irreducible subset of $Spec(E)$. Obviously, $\Im(\mathcal{A}) = \bigcap_{\mathcal{P}\in\mathcal{A}} \mathcal{P}$ is a proper submodule of E and $\mathcal{A} \subseteq V(\Im(\mathcal{A}))$. Now suppose $\mathcal{I}\mathcal{N} \subseteq \Im(\mathcal{A})$, where $\mathcal{I}$ is an ideal of $\mathfrak{R}$ and $\mathcal{N}$ is a submodule of E. By using Proposition 2.9 we get $\mathcal{A} \subseteq V(\mathcal{I}\mathcal{N}) \subseteq V(\mathcal{N}) \cup V(\mathcal{I}E)$. Thus either $\mathcal{A} \subseteq V(\mathcal{N})$ or $\mathcal{A} \subseteq V(\mathcal{I}E)$ because $\mathcal{A}$ is irreducible. Now, if $\mathcal{A} \subseteq V(\mathcal{N})$, then $\mathcal{N} \subseteq \mathcal{P}$, for all $\mathcal{P} \in \mathcal{A}$, i.e., $\mathcal{N} \subseteq \Im(\mathcal{A})$. Also, if $\mathcal{A} \subseteq V(\mathcal{I}E)$, then $\mathcal{I}E \subseteq \mathcal{P}$, for all $\mathcal{P} \in \mathcal{A}$, i.e., $\mathcal{I}E \subseteq \Im(\mathcal{A})$. Therefore by Definition 2.1, $\Im(\mathcal{A})$ is a prime submodule. □

The converse of the above Theorem is not true in general, the following Theorem shows when the converse is true.

Theorem 2.36. [7, Theorem 3.4] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. If $\mathcal{A} \subseteq Spec(E)$ such that $\Im(\mathcal{A})$ is a prime submodule and $\Im(\mathcal{A}) \in \overline{\mathcal{A}}$, then $\mathcal{A}$ is irreducible.

Proof. Assume that $\mathcal{Q} := \Im(\mathcal{A})$ is a prime submodule of E and $\mathcal{Q} \in \overline{\mathcal{A}}$. Now $\mathcal{A} \subseteq V(Q)$, then $\overline{\mathcal{A}} \subseteq V(\mathcal{Q})$ since $V(\mathcal{Q})$ is closed. Also by Proposition 2.32, $V(\mathcal{Q}) \subseteq \overline{\mathcal{A}}$. i.e. $\overline{\mathcal{A}} = V(\mathcal{Q})$. Now suppose $\mathcal{A} \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$, where $\mathcal{A}_1,\ \mathcal{A}_2$ are closed sets in $Spec(E)$. This yields to $\overline{\mathcal{A}} \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$. So $V(\mathcal{Q}) \subseteq \mathcal{A}_1 \cup \mathcal{A}_2$ and by Lemma 2.34, $V(\mathcal{Q})$ is irreducible, $V(\mathcal{Q}) \subseteq \mathcal{A}_1$ or $V(\mathcal{Q}) \subseteq \mathcal{A}_2$. But $\mathcal{A} \subseteq V(\mathcal{Q})$, thus $\mathcal{A} \subseteq \mathcal{A}_1$ or $\mathcal{A} \subseteq \mathcal{A}_2$. Hence $\mathcal{A}$ is irreducible. □

The assumption $\Im(\mathcal{A}) \in \overline{\mathcal{A}}$ is a necessary condition in the above Theorem,

Example 2.37. Let $\mathfrak{R} = \mathbb{Z}$, $E = \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $\mathcal{A} = \{0 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5, \mathbb{Z}_3 \oplus 0 \oplus \mathbb{Z}_5\}$. Now $\Im(\mathcal{A}) = 0 \oplus 0 \oplus \mathbb{Z}_5$ is a prime submodule of E and $\mathcal{A} = V(0 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5) \cup V(\mathbb{Z}_3 \oplus 0 \oplus \mathbb{Z}_5) = \{0 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5\} \cup \{\mathbb{Z}_3 \oplus 0 \oplus \mathbb{Z}_5\}$. Thus $\mathcal{A}$ is not irreducible.

Definition 2.38. Let $\mathfrak{R}$ be a ring, E a left $\mathfrak{R}$-module and $\mathcal{N}$ is a submodule of E.

(1) If there is a prime submodule containing $\mathcal{N}$, then the prime radical of $\mathcal{N}$ [27] is

$$\sqrt{\mathcal{N}} = \bigcap \{\mathcal{P} : \mathcal{P} \text{ is a prime submodule of } E \text{ and } \mathcal{N} \subseteq \mathcal{P}\}.$$

(2) If there is no prime submodule containing $\mathcal{N}$, then we put $\sqrt{\mathcal{N}} = E$.

(3) The prime radical of E is equal to $\sqrt{(0)}$ i.e. $rad_{\mathfrak{R}}(E) = \sqrt{(0)}$.

Corollary 2.39. [7, Corollary 3.6] Let $\mathfrak{R}$ be a ring, E a left $\mathfrak{R}$-module and $\mathcal{N}$ is a submodule of E. The subset $V(\mathcal{N})$ is irreducible in $Spec(E)$ if and only if $\sqrt{\mathcal{N}}$ is a prime submodule of E.

Proof. Suppose that $V(\mathcal{N})$ is an irreducible subset of $Spec(E)$, by Theorem 2.35 $\Im(V(\mathcal{N}))$ is a prime submodule. But $\Im(V(\mathcal{N})) = \sqrt{\mathcal{N}}$. Conversely, let $\sqrt{\mathcal{N}}$ be a prime submodule of $Spec(E)$, Clearly $V(\mathcal{N}) = V(\sqrt{\mathcal{N}})$, i.e. $\sqrt{\mathcal{N}} \in V(\mathcal{N}) = \overline{V(\mathcal{N})}$. Thus by Theorem 2.36, $V(\mathcal{N})$ is irreducible. □

Remark 2.40. [7, Corollary 3.6] From the above Corollary, $Spec(E)$ is irreducible if and only if $\sqrt{(0)} = rad_{\mathfrak{R}}(E)$ is a prime submodule since $Spec(E) = V((0))$.

Let $\mathfrak{R}$ be a ring and E an $\mathfrak{R}$-module. Recall that, E is a *homogeneous semisimple* module if it is a direct sum of isomorphic simple $\mathfrak{R}$-modules, equivalently, $Ann_{\mathfrak{R}}(E)$ is a maximal ideal of $\mathfrak{R}$. Also, if $\mathcal{P}$ is proper submodule of E and the factor module $E/\mathcal{P}$ is a homogeneous semisimple, then $\mathcal{P}$ is called a *virtually maximal* submodule of E. The following Theorems, Theorem 2.41 and Theorem 2.42, give us a characterization for irreducible spectrum over PI-rings or FBN-rings.

Theorem 2.41. [7, Theorem 3.7] Let $\mathfrak{R}$ be a PI-ring (or an FBN-ring) and E a nonzero left $\mathfrak{R}$-module. If $\mathfrak{R}$ is an Artinian ring or E is left Artinian, then the following assertions are equivalent:

(i) $Spec(E)$ is irreducible.

(ii) $E/rad_{\mathfrak{R}}(E)$ is a nonzero homogeneous semisimple module.

(iii) $Spec(E)$ is nonempty and $\forall\, \mathcal{N} \neq E$ submodule of E, either $V(\mathcal{N})$ is empty or $V(\mathcal{N})$ is irreducible.

Proof. $(i) \Rightarrow (ii)$ Suppose that $Spec(E)$ is irreducible. Then $Spec(E) \neq \emptyset$ by the definition of irreducible spaces; and $rad_{\mathfrak{R}}(E)$ is a prime submodule by

Corollary 2.39.

Case (1): If E is a left Artinian module, then $rad_{\mathfrak{R}}(E)$ is a virtually maximal submodule of E; by [5, Corollary 1.6]. Thus, $E/rad_{\mathfrak{R}}(E)$ is a homogeneous semisimple module.

Case (2): If $\mathfrak{R}$ is an Artinian ring and E a left $\mathfrak{R}$-module, then $\mathfrak{R}/\mathcal{P}$ is simple Artinian where $\mathcal{P} := (rad_{\mathfrak{R}}(E) : E)$ is a maximal (prime) ideal. Thus $E/rad_{\mathfrak{R}}(E)$ is a homogeneous semisimple module; $Ann(E/rad_{\mathfrak{R}}(E))$ is maximal ideal.

Finally, $rad_{\mathfrak{R}}(E)$ is a proper submodule of E since $rad_{\mathfrak{R}}(E)$ is prime submodule, and so $E/rad_{\mathfrak{R}}(E)$ is nonzero.

$(ii) \Rightarrow (iii)$ Suppose $E/rad_{\mathfrak{R}}(E)$ is a nonzero homogeneous semisimple module $\mathcal{N}$ is a proper submodule of E. Then either $\sqrt{\mathcal{N}} = E$ or $\sqrt{\mathcal{N}}/rad_{\mathfrak{R}}(E)$ is a proper submodule of $E/rad_{\mathfrak{R}}(E)$ since $rad_{\mathfrak{R}}(E) \subseteq \sqrt{\mathcal{N}}$. Now, we will assume that $\sqrt{\mathcal{N}} \neq E$ because if $\sqrt{\mathcal{N}} = E$, then $V(\mathcal{N}) = \emptyset$. Thus, $\sqrt{\mathcal{N}}/rad_{\mathfrak{R}}(E)$ is a prime submodule of $E/rad_{\mathfrak{R}}(E)$ since $E/rad_{\mathfrak{R}}(E)$ is a homogeneous semisimple module. This means, $\sqrt{\mathcal{N}}$ is a prime submodule of E. Therefore, $V(\mathcal{N})$ is irreducible; by Corollary 2.39.

$(iii) \Rightarrow (i)$ Clear (since $V(0) = Spec(E)$). □

A left $\mathfrak{R}$-module E is called *co-semisimple* module if every proper submodule of E is an intersection of maximal submodules. Thus if E is co-semisimple $\mathfrak{R}$-module and $\mathcal{Q}$ is a proper submodule, then $\mathcal{Q}$ is contained in a maximal submodule. The semisimple modules (direct sum of simple modules) are co-semisimple, see [7].

Theorem 2.42. [7, Theorem 3.7] Let $\mathfrak{R}$ be a PI-ring (or an FBN-ring) and E a nonzero left $\mathfrak{R}$-module. If E is left semisimple module, then the following assertions are equivalent:

(i) $Spec(E)$ is irreducible.

(ii) $E/rad_{\mathfrak{R}}(E)$ is a homogeneous semisimple module.

(iii) $V(\mathcal{N})$ is irreducible, $\forall\, \mathcal{N} \neq E$ submodule of E,

Proof. $(i) \Rightarrow (ii)$ Suppose that $Spec(E)$ is irreducible. Then $Spec(E) \neq \emptyset$ and $rad_{\mathfrak{R}}(E)$ is a prime submodule; by Corollary 2.39. If E is left semisimple module, then by [5, Corollary 1.6], $rad_{\mathfrak{R}}(E)$ is a virtually maximal submodule of E, i.e., $E/rad_{\mathfrak{R}}(E)$ is a homogeneous semisimple module.
$(ii) \Rightarrow (iii)$ Let E be a semisimple module. Let $\mathcal{N}$ be a proper submodule of E, $V(\mathcal{N}) \neq \emptyset$ since every proper submodule of E is contained in a maximal submodule. Clearly $rad_{\mathfrak{R}}(E) \subseteq \sqrt{\mathcal{N}}$. Thus $\sqrt{\mathcal{N}}/rad_{\mathfrak{R}}(E)$ is a proper submodule of $E/rad_{\mathfrak{R}}(E)$. Since $E/rad_{\mathfrak{R}}(E)$ is a homogeneous semisimple module, $\sqrt{\mathcal{N}}/rad_{\mathfrak{R}}(E)$ is a prime submodule of $E/rad_{\mathfrak{R}}(E)$, i.e. $\sqrt{\mathcal{N}}$ is a prime submodule of E. Therefore, $V(\mathcal{N})$ is irreducible, by Corollary 2.39.
$(iii) \Rightarrow (i)$ Clear (since $V(0) = Spec(E)$). □

2.5 Classical Zariski topology in view of spectral spaces

Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. Let $Spec(E)$ be provided with the classical Zariski topology. First, we introduce the definition of a spectral space which defined by Hochster [14] as follows,

Definition 2.43. A topological space T is a *spectral space* if there exits a commutative ring S such that T is homeomorphic to $Spec(S)$, where the topology on $Spec(S)$ is the Zariski topology of rings.

Hochster [14, p.52,Proposition 4] considered another characterization for spectral spaces, the topological space T which satisfy the following conditions :

(1) T is a compact space.

(2) T is a T_0-space.

(3) The compact open subsets of T are closed under finite intersection and form an open basis.

(4) Every irreducible closed subset of T has a generic point.

Let $\mathfrak{R}$ be any ring. In the following Proposition we prove that $Spec(\mathfrak{R})$ is a spectral space.

Proposition 2.44. Let $\mathfrak{R}$ be a ring. Then $Spec(\mathfrak{R})$ is a spectral space.

Proof. Refer to Chapter 1, $Spec(\mathfrak{R})$ is a compact space and the compact open subsets of $Spec(\mathfrak{R})$ are closed under finite intersection and form an open basis, by Proposition 1.10. Now by Theorem 1.11, $Spec(\mathfrak{R})$ is a T_0-space. Finally by Lemma 1.23, every irreducible closed subset of $Spec(\mathfrak{R})$ has a generic point, since the closed set $V(\mathcal{I}) = V(\sqrt{\mathcal{I}})$ is irreducible if and only if $\sqrt{\mathcal{I}}$ is prime ideal, thus $V(\sqrt{I}) = \overline{\{\sqrt{I}\}}$. Therefore, $Spec(\mathfrak{R})$ is a spectral space. □

The following example shows that the compactness property does not hold for all $Spec(E)$.

Example 2.45. [7, Example 2.23] Let E be a $\mathbb{Z}$-module where

$$E = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \cdots \oplus \mathbb{Z}_{p_i} \oplus \cdots$$

where p_i is a non-negative prime number. In Example 2.23 we show that, if $\mathcal{N} = \mathbb{Z}_2 \oplus 0 \oplus 0 \oplus \cdots \leq E$, then $V(\mathcal{N}) = Spec(E) \setminus \{P_1\}$. Thus $V(\mathcal{N})$ is infinite and $V(\mathcal{N}) \neq Spec(E)$. Thus $Spec(E)$ cannot have a finite open cover. Thus it is not compact.

Next Proposition shows that $Spec(E)$ is a T_0-space for any module E and discuss the existence of generic points for irreducible closed sets of $Spec(E)$.

Proposition 2.46. [7, Proposition 3.8] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. Then

(1) $Spec(E)$ is a T_0-space, for any $\mathfrak{R}$-module E.

(2) If $\mathcal{P} \in Spec(E)$, then $\mathcal{P}$ is a generic point of the irreducible closed subset $V(\mathcal{P})$.

(3) If $\mathcal{K}$ is a finite irreducible closed subset of $Spec(E)$, then $\mathcal{K}$ has a generic point.

Proof. (1) A topological space is a T_0-space if and only if the closures of distinct points are distinct. Thus by Corollary 2.33 (i), if $\mathcal{P}_1,\ \mathcal{P}_2 \in Spec(E)$, we have $\overline{\{\mathcal{P}_1\}} = \overline{\{\mathcal{P}_2\}}$ if and only if $\mathcal{P}_1 = \mathcal{P}_2$.
(2) By Corollary 2.33 (i), $V(\mathcal{P}) = \overline{\{\mathcal{P}\}}$ i.e. $\mathcal{P}$ is a generic point for $V(\mathcal{P})$.
(3) Let $\mathcal{K} = \{\mathcal{P}_1, \mathcal{P}_2, \cdots, \mathcal{P}_n\}$ be a finite irreducible closed subset of $Spec(E)$, where $\{\mathcal{P}_i\}_{i=1}^n \in Spec(E)$, $n \in \mathbb{N}$. And by Proposition 2.32, $\mathcal{K} = \overline{\mathcal{K}} = V(\mathcal{P}_1) \cup V(\mathcal{P}_2) \cup \cdots \cup V(\mathcal{P}_n)$. But $\mathcal{K}$ is irreducible subset, hence $\mathcal{K} = V(\mathcal{P}_i)$ for some $1 \leq i \leq n$. Hence, by (2) $\mathcal{K}$ has a generic point. □

Behboodi and Haddadi mentioned in [7] that they didn't find an example of irreducible closed set of $Spec(E)$ that does not have a generic point. Then, they conjectured that, every irreducible closed subset of $Spec(E)$ has a generic point where E is any $\mathfrak{R}$-module. This conjecture was partially solved by Ansari-Toroghy and Ovlyaee-Sarmazdeh in [2, Theorem 3.8]. Before providing the solution of the conjecture, we recall some needed definitions.

Definition 2.47. [1] Let $\mathfrak{R}$ be a commutative ring and E a left $\mathfrak{R}$-module.

(1) Let $P \in Spec(R)$. The set of all P-prime submodules of E is

$$Spec_P(E) = \{\mathcal{P} \in Spec(E) \mid P = (\mathcal{P} : E)\},$$

where $\mathcal{P}$ is the P-prime submodule of E.

(2) The saturation of a submodule $\mathcal{N}$ of E with respect to $P \in Spec(\mathfrak{R})$ is

$$S_P(\mathcal{N}) = \{e \in E \mid te \in \mathcal{N} \text{ for some } t \in \mathfrak{R} \setminus P\}.$$

(3) The torsion submodule of E is

$$T(E) = \{e \in E \mid \exists r \in \mathfrak{R} \setminus 0 : re = 0\}.$$

Remark 2.48. Let $\mathfrak{R}$ be a commutative ring and E a left $\mathfrak{R}$-module. Then

(1) If $T(E) = 0$, then E is called a torsion-free module.

(2) If $T(E) = E$, then E is called a torsion module.

(3) If $P = (0)$ and $\mathcal{N} = (0)$, then $S_{(0)}(0) = T(E)$.

Definition 2.49. [1] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module.

(1) The natural map of a nonempty $Spec(E)$ is defined as follows:

$$\psi : Spec(E) \to Spec(\mathfrak{R}/Ann(E))$$
$$\mathcal{P} \to \overline{(\mathcal{P} : E)} = (\mathcal{P} : E)/Ann(E)$$

(2) If $Spec(E) = \emptyset$ or $Spec(E) \neq \emptyset$ and the natural map of $X = Spec(E)$ is injective (one-to-one), then E is called an X-injective module.

Proposition 2.50. [19, Proposition 3.2] Let $\mathfrak{R}$ be a commutative ring, E an $\mathfrak{R}$-module, $\mathcal{P}, \mathcal{Q} \in Spec(E)$ and $P \in Spec(\mathfrak{R})$. Then the following assertions are equivalent:

1. E is X-injective.

2. If $(\mathcal{P} : E) = (\mathcal{Q} : E)$, then $\mathcal{P} = \mathcal{Q}$.

3. $|Spec_P(E)| \leq 1, \forall P \in Spec(\mathfrak{R})$.

Proof. Suppose E is an X-injective module, i.e $\psi(\mathcal{P}) = \psi(\mathcal{Q})$ implies $\mathcal{P} = \mathcal{Q}$. If $(\mathcal{P} : E) = (\mathcal{Q} : E)$, then $(\mathcal{P} : E)/Ann(E) = (\mathcal{Q} : E)/Ann(E)$ which implies $\mathcal{P} = \mathcal{Q}$. Now suppose (2) holds, if $\mathcal{P}, \mathcal{Q} \in Spec_P(E)$ we have $P = (\mathcal{P} : E) = (\mathcal{Q} : E)$ i.e. $\mathcal{P} = \mathcal{Q}$. Thus $|Spec_P(E)| \leq 1$. Finally suppose (3) holds and for $\mathcal{P}, \mathcal{Q} \in Spec(E)$, $\psi(\mathcal{P}) = \psi(\mathcal{Q})$ implies that $(\mathcal{P} : E) = (\mathcal{Q} : E) = P$. Thus by (3), $\mathcal{P} = \mathcal{Q}$ i.e. ψ is one-to-one. □

Proposition 2.51. [1] Let $\mathfrak{R}$ be a commutative ring and E an $\mathfrak{R}$-module, then

(1) E is X-injective if and only if for every $\mathcal{P} \in Spec(E)$, $\mathcal{P} = S_P(PE)$ for $P = (\mathcal{P} : E)$.

(2) If E is a top module, then E is an X-injective $\mathfrak{R}$-module.

(3) If E is X-injective, then every homomorphic image of E is X-injective.

Proof. (1) Suppose E is X-injective. Let $\mathcal{P} \in Spec(E)$, thus by Proposition 2.50, $Spec_P(E) \neq \emptyset$ where $P = (\mathcal{P} : E)$. Now by [20, Corollary 3.7] ,

$Spec_P(E) \neq \emptyset \Leftrightarrow S_P(PE) \in Spec_P(E)$. Now, $P = (\mathcal{P} : E)$ means $PE \subseteq \mathcal{P} \neq E$, thus $S_P(PE) = S_P(\mathcal{P}) \neq E$. But $S_P(\mathcal{P}) = \{e \in E \mid \exists\, t \in \mathfrak{R} \setminus P \ : \ te \in \mathcal{P}\} = \{e \in E \mid \exists\, t \in \mathfrak{R} \setminus P \ : \ e \in \mathcal{P}\} = \mathcal{P}$ since $\mathcal{P} \in Spec(E)$ i.e. $S_P(PE) = \mathcal{P} \neq E$. Thus $(S_P(PE) : E) = (\mathcal{P} : E)$. Therefore, $\mathcal{P} = S_P(PE)$ since E is X-injective. Conversely, suppose for every $\mathcal{P} \in Spec(E)$, $\mathcal{P} = S_P(PE)$ for $P = (\mathcal{P} : E)$. Let $\mathcal{P}, \mathcal{Q} \in Spec(E)$ such that $(\mathcal{P} : E) = (\mathcal{Q} : E)$. Thus $\mathcal{P} = S_P(PE)$ and $\mathcal{Q} = S_Q(QE)$ where $P = (\mathcal{P} : E) = (\mathcal{Q} : E) = Q$. Hence, $\mathcal{P} = \mathcal{Q}$. Therefore by Proposition 2.50, E is X-injective.

(2) By [26, Theorem 3.5] E is top module implies $|Spec_P(E)| \leq 1$ for every $P \in Spec(\mathfrak{R})$ which equivalent to E is X-injective by Proposition 2.50.

(3) Suppose E is X-injective. By Proposition 2.50 for every $P \in Spec(\mathfrak{R})$, $|Spec_P(E)| \leq 1$. The result follows since for every submodule $\mathcal{N}$ of E, $Spec(E/\mathcal{N}) = \{\mathcal{P}/\mathcal{N} \mid \mathcal{P} \in Spec(E) : \mathcal{N} \subseteq \mathcal{P}\}$, so for every $P \in Spec(\mathfrak{R})$, $Spec_P(E/\mathcal{N}) = \{\mathcal{P}/\mathcal{N} \in Spec(E/\mathcal{N}) | P = (\mathcal{P}/\mathcal{N} : E/\mathcal{N}) = (\mathcal{P} : E)\}$. Thus $|Spec_P(E/\mathcal{N})| \leq 1$. i.e. $E/\mathcal{N}$ is X-injective. □

Next we consider a needed Proposition. For it's proof see [1].

Proposition 2.52. [1, Corollary 3.19] Let $\mathfrak{R}$ be a PID and the $\mathfrak{R}$-module E be a torsion-free or torsion-module, then E is top module if and only if E is X-injective.

Theorem 2.53. [2, Theorem 3.8] Let $\mathfrak{R}$ be a PID and E an X-injective left $\mathfrak{R}$-module. If $\mathcal{A}$ is an irreducible closed subset of $Spec(E)$, then $\mathcal{A}$ has a generic point.

Proof. Let $\mathcal{A} \subseteq Spec(E)$ be an irreducible closed subset. Then we have two cases:

Case (1): If E is a top module, then $\mathcal{A} = V(\mathcal{N}) = V(\sqrt{\mathcal{N}})$ for some submodule $\mathcal{N}$ of E. By Corollary 2.39, $\sqrt{\mathcal{N}}$ is a prime submodule of E since $\mathcal{A} = V(\sqrt{\mathcal{N}})$ is irreducible. Hence, $\mathcal{A}$ has a generic point.

Case (2): If E is not a top module, let $0 \neq S_{(0)}(0) \subset E$ since otherwise i.e if $S_{(0)}(0) = 0$ or $S_{(0)}(0) = E$, then E is a top module by [1, Corollary 3.19]. Now, suppose $\mathcal{A}$ is infinite subset, because if $\mathcal{A}$ is finite, then it has a generic point by Proposition 2.46. Now by Theorem 2.35, $\Im(\mathcal{A})$ is a prime submodule of E. Thus by Proposition 2.51, $\Im(\mathcal{A}) = S_P(PE)$ where $P \in Spec(\mathfrak{R})$. Now, $P = (0)$ since if not, there is a maximal ideal $Q \neq P$ of $\mathfrak{R}$ such that $S_Q(QE) = S_P(PE)$, and this contradicts $Q \neq P$. Therefore, $\Im(\mathcal{A}) = S_{(0)}(0)$ and if $\mathcal{P} \in \mathcal{A}$, then $S_{(0)}(0) \subseteq \mathcal{P}$. Let $\widehat{E} = E/S_{(0)}(0)$, $\widehat{X} = Spec(E/S_{(0)}(0)) = \{\mathcal{P}/S_{(0)}(0) | \mathcal{P} \in Spec(E)\}$ and $\widehat{\mathcal{A}} = \{\widehat{\mathcal{P}} = \mathcal{P}/S_{(0)}(0) | \mathcal{P} \in \mathcal{A}\}$. Now, $\widehat{E}$ is torsion-free, since $S_{(0)}(0) = T(E)$ and $E/T(E)$ is torsion-free. Also, by Proposition 2.51, $\widehat{E}$ is X-injective, since E is X-injective. Therefore, $\widehat{E}$ is a top module, by Proposition 2.52. Now $\widehat{X}$ has Zariski topology Definition 2.7, $X = Spec(E)$ has classical Zariski topology and define the map $\psi : \widehat{X} \to X$ as follows $\psi(\mathcal{P}/S_{(0)}(0)) = \mathcal{P}$. Clearly, ψ is a one-to-one map. Let $\mathcal{L}_{i,j}$ be a submodule of E and $\bigcap_{i \in I}(\bigcup_{j=1}^{m_i} V(\mathcal{L}_{i,j}))$ be a closed set of $Spec(E)$. Then ψ is continuous map since

$$\psi^{-1}\left(\bigcap_{i \in I}\left(\bigcup_{j=1}^{m_i} V\left(\mathcal{L}_{i,j}\right)\right)\right) = \bigcap_{i \in I}\left(\bigcup_{j=1}^{m_i} V(\widehat{\mathcal{L}_{i,j}})\right).$$

Moreover, ψ is a closed map since for any submodule $\mathcal{L}$ of $\widehat{E}$ we have $\psi(V(\widehat{\mathcal{L}})) =$

$V(\mathcal{L})$. Therefore, $\widehat{\mathcal{A}} = \psi^{-1}(\mathcal{A})$ is an irreducible closed set in $\widehat{X}$. And since $\widehat{E}$ is a top module, $\widehat{\mathcal{A}}$ has a generic point. So there is a prime submodule $\widehat{\mathcal{P}}$ of $\widehat{E}$ such that $\widehat{\mathcal{A}} = V(\widehat{\mathcal{P}})$. Therefore $\mathcal{A}$ has a generic point since $\mathcal{A} = \psi(\widehat{\mathcal{A}}) = \psi(V(\widehat{\mathcal{P}})) = V(\mathcal{P})$. □

The following Theorem shows that, if the $\mathfrak{R}$-module E has finite prime spectrum, then $Spec(E)$ is a spectral space.

Theorem 2.54. [7, Theorem 3.9] Let $\mathfrak{R}$ be a ring and E a left $\mathfrak{R}$-module. If $Spec(E)$ is a finite set, then $Spec(E)$ is a spectral space.

Proof. Suppose that $Spec(E)$ is a finite set. By Proposition 2.46, $Spec(E)$ is a T_0-space and every irreducible closed subset of $Spec(E)$ has a generic point. Also, $Spec(E)$ is finite implies that it is a compact space and the compact open subsets of $Spec(E)$ are closed under finite intersection and form an open basis. Therefore, $Spec(E)$ satisfies the conditions in the Hochster's characterization of a spectral space. i.e. $Spec(E)$ is a spectral space. □

Corollary 2.55. Let $\mathfrak{R}$ be a ring and E a finite left $\mathfrak{R}$-module, then $Spec(E)$ is a spectral space.

Proof. Suppose E is finite $\mathfrak{R}$-module. Then $Spec(E)$ must be finite. Thus it is a spectral space. □

Next Proposition talks about the relation between Noetherian modules and Noetherian spaces if the module is a top module.

Proposition 2.56. Let $\mathfrak{R}$ be a ring and E a Noetheian $\mathfrak{R}$-module. If E is a top module, then $Spec(E)$ is a Noetherian space.

Proof. Suppose E is a top module. Let $(V(\mathcal{N}_i))_{i\in\mathbb{N}}$ be an infinite collection of closed sets of $Spec(E)$ such that $V(\mathcal{N}_{i+1}) \subseteq V(\mathcal{N}_i)$. Since $V(\mathcal{N}) = V(\sqrt{\mathcal{N}})$ for any submodule of E, we have $\sqrt{\mathcal{N}_i} \subseteq \sqrt{\mathcal{N}_{i+1}}$ for all i. Hence we have an ascending chain of submodules of E and since E is a Noetherian ring there is n such that $\sqrt{\mathcal{N}_i} = \sqrt{\mathcal{N}_n}$ for all $i \geq n$. Therefore, $V(\sqrt{\mathcal{N}_i}) = V(\sqrt{\mathcal{N}_n})$. i.e. $Spec(E)$ is a Noetherian space. □

The following Theorem discuss when the X-injective module over PID is a spectral space.

Theorem 2.57. [2, Theorem 3.8] Let $\mathfrak{R}$ be a PID and E an X-injective $\mathfrak{R}$-module. If $Spec(E)$ is a Noetherian space, then $Spec(E)$ is a spectral space.

Proof. Suppose $Spec(E)$ is a Noetherian space, then it is a compact space and every open subset is compact, thus compact open subsets are closed under finite intersection and form an open basis. Now $Spec(E)$ is a T_0-space by Proposition 2.46. Finally by Theorem 2.53 every irreducible closed subset of $Spec(E)$ has a generic point. i.e. $Spec(E)$ is a spectral space. □

In [19] Lu considered another Zariski topology on $Spec(E)$ such that the closed sets are $V^*(\mathcal{N}) = \{\mathcal{P} \in Spec(E) | (\mathcal{N} : E) \subseteq (\mathcal{P} : E)\}$ and studied it in view of spectral spaces. The next Proposition compares between this topology and the classical Zariski topology.

Proposition 2.58. [7, Proposition 3.11] Let $\mathfrak{R}$ be a commutative ring and E be a finitely generated $\mathfrak{R}$-module. Then the classical Zariski topology of E and the Zariski topology of E considered in [19], coincide if and only if E is a multiplication module.

Proof. Assume that the classical Zariski topology of E and the Zariski topology of E considered in [19], coincide. Then by Proposition 2.46, $Spec(E)$ with the topology considered in [19] is a T_0-space. Now by [19, Corollary 6.6], E is a multiplication module. The converse is evident. □

Corollary 2.59. [7, Corollary 3.12] Let $\mathfrak{R}$ be a commutative ring. If E is a finitely generated multiplication $\mathfrak{R}$-module, then $Spec(E)$ under classical Zariski topology is a spectral space.

Proof. By Proposition 2.58, the classical Zariski topology of E is coincide with the Zariski topology of E considered in [19], then by [19, Corollary 6.6] $Spec(E)$ under the Lu Zariski topology is a spectral space. □

References

[1] H. Ansari-Toroghy and R. Ovlyaee-Sarmazdeh. ***On the prime spectrum of X-injective modules***. **Communications in Algebra** , 38(7):2606–2621, 2010.

[2] H. Ansari-Toroghy and R. Ovlyaee-Sarmazdeh. ***On the prime spectrum of a module and Zariski topologies***. **Communications in Algebra**, 38 (12):4461–4475, 2010.

[3] Michael F. Atiyah and I.G. MacDonald. **Introduction to Commutative Algebra**. Addison-Wesley, Reading, Massachusetts, 1969.

[4] Igor R. Shafarevich (auth.). **Basic Algebraic Geometry 1: Varieties in Projective Space.** Springer-Verlag Berlin Heidelberg, 3rd edition, 2013.

[5] M. Behboodi. ***A generalization of the classical krull dimension for modules***. **Journal of Algebra**, 305(2):1128–1148, 2006.

[6] M. Behboodi and M.R. Haddadi. ***Classical Zariski topology of modules and spectral spaces II***. **International Electronic Journal of Algebra**, 4 (4):131–148, 2008.

[7] M. Behboodi and M.R. Haddadi. ***Classical Zariski topology of modules and spectral spaces I***. **International Electronic Journal of Algebra**, 4 (4):104–130, 2008.

[8] M. Behboodi, O.A.S. Karamzadeh, and H. Koohy. ***Modules whose certain submodules are prime***. **Vietnam J. Math**, 32(3):303–317, 2004.

[9] Nicolas Bourbaki. **Commutative Algebra**. Addison-Wesley, 1972.

[10] Janko Bracic. ***The prime spectrum of a module***. **Functional analysis IX**, page 32, 2007.

[11] John Dauns. ***Prime modules***. **Journal fór Mathematik. Band**, 298:21, 1978.

[12] Kent R. Fuller (auth.) Frank W. Anderson. **Rings and Categories of Modules**. Graduate Texts in Mathematics 13. Springer New York, 2nd edition, 1974.

[13] Kenneth R Goodearl and Robert Breckenridge Warfield Jr. **An introduction to noncommutative Noetherian rings**, volume 61. Cambridge university press, 2004.

[14] Melvin Hochster. ***Prime ideal structure in commutative rings***. **Transactions of the American Mathematical Society**, 142:43–60, 1969.

[15] Thomas W. Hungerford. **Algebra**. Graduate Texts in Mathematics v. 73. Springer, 1980.

[16] Peter T. Johnstone. **Stone Spaces**. Cambridge studies in advanced mathematics 3. Cambridge University Press, 1st edition, 1982.

[17] Paul E. Long. **An introduction to general topology**. Merrill Publishing Company, 1971.

[18] Chin-Pi Lu. ***Spectra of modules***. **Communications in Algebra**, 23(10): 3741–3752, 1995.

[19] Chin-Pi Lu. ***The Zariski topology on the prime spectrum of a module***. **Houston J. Math**, 25(3):417–432, 1999.

[20] Chin-Pi Lu. ***Saturations of submodules***. **Communications in Algebra**, 31(6):2655–2673, 2003.

[21] Chin-Pi Lu et al. ***Prime submodules of modules***. **Rikkyo Daigaku sugaku zasshi**, 33(1):61–69, 1984.

[22] Shing Hing Man. ***On commutative Noetherian rings which have the spar property***. **Archiv der Mathematik**, 70(1):31–40, 1998.

[23] Shing Hing Man and Patrick F. Smith. ***On chains of prime submodules***. **Israel Journal of Mathematics**, 127(1):131–155, 2002.

[24] Agustin Marcelo and J Munoz Masque. ***Prime submodules, the descent invariant, and modules of finite length***. **Journal of Algebra**, 189(2): 273–293, 1997.

[25] R.L. McCasland and P.F. Smith. ***Prime submodules of noetherian modules***. **The Rocky Mountain Journal of Mathematics**, 23(3):1041–1062, 1993.

[26] R.L. McCasland, M.E. Moore, and P.F. Smith. ***On the spectrum of a module over a commutative ring***. **Communications in Algebra**, 25(1): 79–103, 1997.

[27] Roy L. McCasland and Marion E. Moore. ***On radicals of submodules of finitely generated modules***. **Canadian Mathematical Bulletin**, 29(1): 37–39, 1986.

[28] John C. McConnell, James Christopher Robson, and Lance W. Small. **Noncommutative Noetherian rings**, volume 30. American Mathematical Soc., 2001.

[29] James R. Munkres. **Topology**. Prentice Hall, Inc, 2nd edition, 2000.

[30] Dilek Pusat-Yilmaz and Patrick F. Smith. ***Chain conditions in modules with krull dimension***. **Communications in Algebra**, 24(13):4123–4133, 1996.

[31] R.Y. Sharp. **Steps in Commutative Algebra**. London Mathematical Society St. Cambridge University Press, 2000.

[32] Guoyin Zhang, Fanggui Wang, and Wenting Tong. ***Multiplication modules in which every prime submodule is contained in a unique maximal submodule***. **Communications in Algebra**, 32(5):1945–1959, 2004.

جامعة النجاح الوطنية
كلية الدراسات العليا

دراسة لتبولوجية زاريسكي على الحلقيات بين النظرية والتطبيق

إعداد
نهيل صابر محمد غيث

إشراف
د. خالد عداربة

قدمت هذه الأطروحة استكمالا لمتطلبات الحصول على درجة الماجستير في الرياضيات من كلية الدراسات العليا في جامعة النجاح الوطنية في نابلس–فلسطين.

2021

دراسة لتبولوجية زاريسكي على الحلقيات بين النظرية و التطبيق

إعداد

نهيل صابر محمد غيث

إشراف

د. خالد عداربة

الملخص

تساهم هذه الأطروحة في دراسة توبولوجية زاريسكي على الحلقات بالإضافة إلى تعميمها على الحلقيات. يقدم الجزء الأول من الأطروحة توبولوجية زاريسكي على الحلقات، وتناقش العديد من الخصائص التوبولوجية، على سبيل المثال الفضاءات المحكمة (المتراصة)، مسلمات الإنفصال والفضاءات النوثيرانية.

في الجزء الثاني من الأطروحة، ندرس تعميما لتوبولوجية زاريسكي من الحلقات إلى الحلقيات. تم تقديم هذا التعميم من قبل محمد بهبودي ومحمد حدادي في عام 2008. كما في الجزء الأول، يتم تقديم العديد من ميزات التوبولوجيا كتعميم لخصائص مماثلة في الجزء الأول.

يربط الموضوع من خلال الجزأين الخصائص التوبولوجية بالخصائص الجبرية. وتجدر الإشارة إلى أنه يتم استخدام نتائج الجزأين لبناء أمثلة توضيحية للتوبولوجيا.